

On higher genus Welschinger invariants of del Pezzo surfaces

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Abstract

The Welschinger invariants of real rational algebraic surfaces count real rational curves which represent a given divisor class and pass through a generic conjugation-invariant configuration of points. No invariants counting real curves of positive genera are known in general. We indicate particular situations, when Welschinger-type invariants counting real curves of positive genera can be defined. We also prove the positivity and give asymptotic estimates for such Welschinger-type invariants for several del Pezzo surfaces of degree ≥ 2 and suitable real nef and big divisor classes. In particular, this yields the existence of real curves of given genus and of given divisor class passing through any appropriate configuration of real points on the given surface.

1 Introduction

Welschinger invariants serve as genus zero open Gromov-Witten invariants. For real rational symplectic manifolds [27, 28], they count real rational pseudo-holomorphic curves, realizing a given homology class, passing through a generic conjugation-invariant configuration of points, and equipped with weights ± 1 . In the case of real del Pezzo surfaces, Welschinger invariants count real algebraic rational curves. A more general approach used by J. Solomon allowed him to define also invariants that count real curves of positive genera with fixed complex and real structure of the normalization [25, Theorem 1.3]. However, so far, no any general invariant way to count real curves of positive genera (without fixing their complex and real structure) has been found. In particular, it follows from [13, Theorem 3.1], that if we do not fix complex and real structure of the normalization, then even the count of real plane elliptic curves of any degree ≥ 4 equipped with Welschinger signs is *not invariant* of the choice of the point constraints.

The main goal of this note is to indicate situations, in which the “bad” bifurcation of type [13, Theorem 3.1] does not occur and in which Welschinger-type invariants of positive genera can be defined. So, in Section 2 we introduce higher genus invariants of real del Pezzo surfaces with a disconnected real point set and prove that they indeed do not depend on the choice of point constraints and on variation of the surface. In Section 3 we compute new invariants in several examples

and exhibit a series of invariants, which are positive and are asymptotically comparable with Gromov-Witten invariants. In particular, this yields the existence of real curves of given genus and of given divisor class passing through any appropriate configuration of real points on the given surface.

It is worth mentioning that [15, Theorem 1] states that the count of tropical curves of any genus with appropriate tropical Welschinger signs is invariant of the choice of tropical point constraints for any toric surface. The reason why the “bad” bifurcation does not appear in the tropical limit is discussed in [19, 20].

In our consideration we intensively use techniques of [16] and [17]; for the reader’s convenience, in Appendices 1 and 2, we present needed statements from these works in the form applicable to curves of arbitrary genus.

2 Invariant count of real curves of positive genera

Let X be a real del Pezzo surface with a nonempty real point set $\mathbb{R}X$. Denote by $\text{Pic}^{\mathbb{R}}(X) \subset \text{Pic}(X)$ the subgroup of real divisor classes. For any connected component $G \subset \mathbb{R}X$, one can define a homomorphism $\text{bh}_G : \text{Pic}^{\mathbb{R}}(X) \rightarrow H_1(G; \mathbb{Z}/2)$ (cf. [3]), which sends an effective divisor class $D \in \text{Pic}^{\mathbb{R}}(X)$ to the class $[\mathbb{R}C \cap G] \in H_1(G; \mathbb{Z}/2)$, where $C \in |D|$ is any real curve. Indeed, it can be viewed as the composition of the homomorphisms

$$H_2^{\text{conj}}(X) \rightarrow H_2(X/\text{conj}, \mathbb{R}X; \mathbb{Z}/2) \rightarrow H_1(\mathbb{R}X; \mathbb{Z}/2) \rightarrow H_1(G; \mathbb{Z}/2)$$

given by $[\sigma] \mapsto [\sigma/\text{conj}] \mapsto [\partial(\sigma/\text{conj})] \mapsto [(\partial(\sigma/\text{conj})) \cap G]$. It follows that, for each $D \in \text{Pic}^{\mathbb{R}}(X)$, there is a well-defined value $(\text{bh}_G(D))^2 \in \{0, 1\}$.

Suppose that $\mathbb{R}X$ contains at least $g + 1$ connected components F_0, \dots, F_g for some $g \geq 1$. Put $\widehat{F} = F_0 \cup \dots \cup F_g$, $\underline{F} = (F_0, \dots, F_g)$. We say that a divisor class $D \in \text{Pic}^{\mathbb{R}}(X)$ is \widehat{F} -compatible, if, for any connected component $G \subset \mathbb{R}X \setminus \widehat{F}$, one has $\text{bh}_G(D) = 0$. Note that \widehat{F} -compatible divisor classes $D \in \text{Pic}^{\mathbb{R}}(X)$ satisfy

$$DK_X \equiv \sum_{i=0}^g (\text{bh}_{F_i}(D))^2 \pmod{2}.$$

For any tuple (r_0, \dots, r_g, m) of nonnegative integers, introduce the space $\mathcal{P}_{\underline{r}, m}(X, \underline{F})$ (where $\underline{r} = (r_0, \dots, r_g)$) of configurations of $r_0 + \dots + r_g + 2m$ distinct points of X such that r_i of them belong to F_i , $i = 0, \dots, g$, and the others form m complex conjugate pairs.

Choose any conjugation invariant class $\varphi \in H_2(X \setminus \widehat{F}; \mathbb{Z}/2)$ and pick a big and nef, \widehat{F} -compatible divisor class $D \in \text{Pic}^{\mathbb{R}}(X)$ such that

$$p_a(D) = (D^2 + DK_X)/2 + 1 \geq g \quad \text{and} \quad -DK_X \geq g + 1 - \sum_{i=0}^g (\text{bh}_{F_i}(D))^2. \quad (1)$$

Then there exist nonnegative integers r_0, \dots, r_g, m such that

$$r_0 + \dots + r_g + 2m = -DK_X + g - 1, \quad r_i \equiv (\text{bh}_{F_i}(D))^2 + 1 \pmod{2}, \quad i = 0, \dots, g. \quad (2)$$

If X is sufficiently generic in its deformation class, and $\mathbf{w} \in \mathcal{P}_{\underline{r},m}(X, \underline{F})$ is generic, then the set $\mathcal{C}_g^{\mathbb{R}}(X, D, \mathbf{w})$ of real irreducible curve $C \in |D|$ of genus g , passing through \mathbf{w} , is finite and consists of only immersed curves (see Lemma 15). Furthermore, each curve $C \in \mathcal{C}_g^{\mathbb{R}}(X, D, \mathbf{w})$ has a one-dimensional real branch in each of the components F_0, \dots, F_g of $\mathbb{R}X$. In particular, this yields that $C \setminus \mathbb{R}C$ consists of two connected complex conjugate components, and we denote one of them by $C_{1/2}$. For any vector $\underline{\varepsilon} = (\varepsilon_0, \dots, \varepsilon_g)$ with $\varepsilon_i = \pm 1$, $i = 0, \dots, g$, put

$$W_{g,\underline{r}}(X, D, \underline{F}, \underline{\varepsilon}, \varphi, \mathbf{w}) = \sum_{C \in \mathcal{C}_g^{\mathbb{R}}(X, D, \mathbf{w})} (-1)^{s(C; \underline{F}, \underline{\varepsilon}) + C_{1/2} \circ \varphi},$$

where $s(C, \underline{F}, \underline{\varepsilon})$ is defined as follows: if C is nodal, then this is the number of those real nodes of C in \widehat{F} , which in F_i are represented in real local coordinates as $x^2 + \varepsilon_i y^2 = 0$, $i = 0, \dots, g$ (a node of type $x^2 + y^2 = 0$ is called *solitary*, and of type $x^2 - y^2 = 0$ - *non-solitary*), and if C is not nodal, we locally deform each germ (C, z) , $z \in \text{Sing}(C)$, moving its components to a general position in an equivariant way, and then count real nodes as in the nodal case. Since C is immersed, the number $s(C, \underline{F}, \underline{\varepsilon}) \bmod 2$ does not depend on the choice of local deformations of C . Our main result is the following analog of Welschinger's theorem [27, 28] (see also [16]).

Theorem 1 *Let X be a real del Pezzo surface, F_0, \dots, F_g connected components of $\mathbb{R}X$ for some $g \geq 1$, $D \in \text{Pic}^{\mathbb{R}}(X)$ a nef and big, \widehat{F} -compatible divisor class satisfying (1), r_0, \dots, r_g, m nonnegative integers satisfying (2), and $\varphi \in H_2(X \setminus \widehat{F}; \mathbb{Z}/2)$ a conjugation-invariant class. Let $\underline{\varepsilon} = (\varepsilon_0, \dots, \varepsilon_g)$, $\varepsilon_i = \pm 1$, $i = 0, \dots, g$. Then the following hold.*

(1) *The numbers $W_{g,\underline{r}}(X, D, \underline{F}, \underline{\varepsilon}, \varphi, \mathbf{w})$ do not depend on the choice of a generic configuration $\mathbf{w} \in \mathcal{P}_{\underline{r},m}(X, \underline{F})$ (which further on will be omitted in the notation).*

(2) *If tuples $(X, D, \underline{F}, \varphi)$ and $(X', D', \underline{F}', \varphi')$ are deformation equivalent, then*

$$W_{g,\underline{r}}(X, D, \underline{F}, \underline{\varepsilon}, \varphi) = W_{g,\underline{r}}(X', D', \underline{F}', \underline{\varepsilon}, \varphi').$$

Corollary 1 *Under the hypotheses of Theorem 1, for any generic configuration $\mathbf{w} \in \mathcal{P}_{\underline{r},m}(X, \underline{F})$,*

$$|W_{g,\underline{r}}(X, D, \underline{F}, \underline{\varepsilon}, \varphi)| \leq \#\mathcal{C}_g^{\mathbb{R}}(X, D, \mathbf{w}) \leq GW_g(X, D),$$

where GW_g is the genus g Gromov-Witten invariant. In particular, if $W_{g,\underline{r}}(X, D, \underline{F}, \underline{\varepsilon}, \varphi) \neq 0$, then through any generic configuration $\mathbf{w} \in \mathcal{P}_{\underline{r},m}(X, \underline{F})$, one can trace a real curve $C \in |D|$ of genus g .

In Section 3, we exhibit several examples in which our invariants do not vanish, and therefore prove the existence of real curves of positive genera passing through prescribed configurations of real points.

Proof of Theorem 1. The proof follows the lines of [16], where the case of rational curves has been treated in full detail in the algebraic geometry framework. We only indicate principal points of the argument, referring to Appendix 1, which contains all needed statements from [16].

The strategy of the proof is to verify that the studied enumerative numbers remain constant in general variations of the point constraints $\mathbf{w} \in \mathcal{P}_{\underline{r},m}(X, \underline{F})$ and of the surface X .

Let us fix a surface X and consider the space $\mathcal{P}_n^{\mathbb{C}}(X)$ of n -tuples of distinct points of X . Let $n = r_0 + \dots + r_g + 2m = -DK_X + g - 1$. Then $\mathcal{P}_{\underline{r},m}(X, \underline{F}) \subset \mathcal{P}_n^{\mathbb{C}}(X)$. Introduce the characteristic variety

$$\text{Ch}_n^{\mathbb{C}}(X, D) = \left\{ \mathbf{w} \in \mathcal{P}_n^{\mathbb{C}}(X) \left| \begin{array}{l} \text{there exists a Riemann surface } S_g \text{ of genus } g, \\ \text{an immersion } \nu : S_g \rightarrow X \text{ and an } n\text{-tuple } \mathbf{p} \text{ of distinct points of } S_g \\ \text{such that } \nu(\mathbf{p}) = \mathbf{w}, \nu(S_g) \in |D|, \text{ and } h^1(S_g, \mathcal{N}_{S_g}^{\nu}(-\mathbf{p})) > 0 \end{array} \right. \right\},$$

where $\mathcal{N}_{S_g}^{\nu} = \nu^* \mathcal{T}X / \mathcal{T}S_g$ is the normal bundle. If $p_a(D) > g$, this is a hypersurface in $\mathcal{P}_n^{\mathbb{C}}(X)$. As pointed out in [13, Theorem 3.1], the invariance of Welschinger numbers fails when the (moving) configuration \mathbf{w} hits $\text{Ch}_n^{\mathbb{C}}(X, D)$. Our key observation is that this event does not happen in our situation.

Lemma 1 *Under the hypotheses of Theorem 1, $\mathcal{P}_{\underline{r},m}(X, \underline{F}) \cap \text{Ch}_n^{\mathbb{C}}(X, D) = \emptyset$.*

Proof. Let $\nu : S_g \rightarrow X$ be a conjugation-invariant immersion, $\mathbf{p} \subset S_g$ be a conjugation-invariant n -tuple such that $C = \nu^* S_g \in \mathcal{C}_g^{\mathbb{R}}(X, D, \mathbf{w})$, where $\mathbf{w} = \nu(\mathbf{p}) \in \mathcal{P}_{\underline{r},m}(X, \underline{F})$. Suppose that $h^1(S_g, \mathcal{N}_{S_g}^{\nu}(-\mathbf{p})) > 0$. Then by Riemann-Roch $h^0(S_g, \mathcal{N}_{S_g}^{\nu}(-\mathbf{p})) > 0$. It is well known that $\nu_* \mathcal{N}_{S_g}^{\nu} = \mathcal{J}_C^{\text{cond}} \otimes \mathcal{O}_X(D)$, where $\mathcal{J}_C^{\text{cond}} = \text{Ann}(\overline{\mathcal{O}}_C / \mathcal{O}_C)$ is the conductor ideal sheaf on C (see details, e.g., in [9, Section 4.2.4]). Hence

$$H^0(C, \mathcal{J}_C^{\text{cond}}(-\mathbf{w}) \otimes \mathcal{O}_X(D)) \simeq H^0(S_g, \mathcal{N}_{S_g}^{\nu}(-\mathbf{p})) \neq 0$$

(in the case of $w = z \in \text{Sing}(C)$ for some point $w \in \mathbf{w}$, we define the twisted sheaf $\mathcal{J}_C^{\text{cond}}(-\mathbf{w})$ as the limit when w specializes to the point z along a component of the germ (C, z)). A real nonzero element of $H^0(C, \mathcal{J}_C^{\text{cond}}(-\mathbf{w}) \otimes \mathcal{O}_X(D))$ defines a real curve $C' \neq C$ in the linear system $|D|$, which intersects C at each singular point $z \in \text{Sing}(C)$ with multiplicity $\geq \delta(C, z)$ (see, [9, Section 4.2.4]) and in \mathbf{w} . In view of congruence (2), C' must intersect C in (at least) one additional point in each component F_0, \dots, F_g , and hence

$$\begin{aligned} CC' &\geq \sum_{i=0}^g (r_i + 1) + 2m + 2\delta(C) \\ &= (-DK_X + g - 1) + (g + 1) + (D^2 + DK_X + 2 - 2g) = D^2 + 2, \quad (3) \end{aligned}$$

which is a contradiction. ■

By Lemmas 14-17, in a general smooth equivariant deformation \mathbf{w}_t , $t \in [0, 1]$, of $\mathbf{w} = \mathbf{w}_0$ in $\mathcal{P}_{\underline{r},m}(X, \underline{F})$, for t in the complement to a finite set $\Phi \subset [0, 1]$, the curve collection $\mathcal{C}_g^{\mathbb{R}}(X, D, \mathbf{w})$ consists of immersed Riemann surfaces of genus g , and the values $t \in \Phi$ correspond to degeneration of some curves of $\mathcal{C}_g^{\mathbb{R}}(X, D, \mathbf{w})$ either into nonimmersed, birational images of Riemann surfaces of genus g , or into curves listed in Lemma 17. Lemmas 1 and 18 yield that the numbers $W_{g,\underline{r}}(X, D, \underline{F}, \underline{\varepsilon}, \varphi, \mathbf{w})$

do not change as t varies along any of the components of $[0, 1] \setminus \Phi$. Next, we can suppose that the configuration \mathbf{w}_t is in general position on each of the finitely many curves $C = \nu(\hat{C})$, where $[\nu : \hat{C} \rightarrow X, \mathbf{p}] \in \mathcal{C}_g^{\mathbb{R}}(X, D, \mathbf{w}_t)$, $t \in \Phi$. Then the constancy of the numbers $W_{g,\underline{x}}(X, D, \underline{F}, \underline{\varepsilon}, \varphi, \mathbf{w})$ in these bifurcations follows from Lemmas 18, 22, and 23.

The proof of statement (2) of Theorem 1 amounts in the verification of the constancy of the number $W_{g,\underline{x}}(X, D, \underline{F}, \underline{\varepsilon}, \varphi, \mathbf{w})$ when X smoothly bifurcates through a uninodal del Pezzo surface (see Section 4.2 in Appendix 1) The treatment is based on the use of an appropriate real version of the Abramovich-Bertram-Vakil formula [1], [26, Theorem 4.2], and it literally coincides with that in [17, Section 4], while the key points in this consideration are Lemmas 15(2ii) and 19. \blacksquare

3 Examples

3.1 Small divisors

Proposition 2 *Suppose that the data $X, g, \underline{F}, D, \underline{x}, \varphi$ satisfy the hypotheses of Theorem 1.*

(1) *If $p_a(D) = g$, then $W_{g,\underline{x}}(X, D, \underline{F}, \underline{\varepsilon}, \varphi) = (-1)^{C_{1/2} \circ \varphi}$, where C is any smooth curve from $|D|$.*

(2) *If $p_a(D) = g + 1$, then*

$$W_{g,\underline{x}}(X, D, \underline{F}, \underline{\varepsilon}, \varphi) = \begin{cases} \sum_{i=0}^g \varepsilon_i(r_i + 1 - \chi(F_i)), & \text{if } \varphi = 0, \\ \sum_{i=0}^g \varepsilon_i(r_i + 1 - \chi(F_i)) - \chi(\mathbb{R}X \setminus \widehat{F}), & \text{if } \varphi = [\mathbb{R}X \setminus \widehat{F}] \end{cases}$$

Proof. The first formula is evident, since the point constraints define a unique smooth curve. In the second case, the point constraints define a pencil of curves in $|D|$, which by Bézout's argument similar to (3) have, additionally to \mathbf{w} , an extra common point in each component F_0, \dots, F_g , and hence the result follows from the Morse formula after blowing up of all $\sum_{i=0}^g (r_i + 1)$ real common points of the pencil. \blacksquare

Example 3 *Suppose that X is a two-component real cubic surface in \mathbb{P}^3 , $F_0 \simeq \mathbb{R}P^2$, $F_1 \simeq S^2$, and let $g = 1$. Then (see [21]) X contains precisely three real (-1) -curves E_1, E_2, E_3 such that $\mathbb{R}E_1 \cup \mathbb{R}E_2 \cup \mathbb{R}E_3 \subset F_0$, and each real affective, big and nef divisor can be represented as $D = m_1E_1 + m_2E_2 + m_3E_3$ with $0 < 2m_i \leq m_1 + m_2 + m_3$, $i = 1, 2, 3$. In particular, $-K_X = E_1 + E_2 + E_3$. Since $p_a(-2K_X - E_i) = 2$, $i = 1, 2, 3$, we have*

$$W_{1,\underline{x}}(X, -2K_X - E_i, \underline{F}, (\varepsilon_0, \varepsilon_1), 0) = \varepsilon_0 r_0 + \varepsilon_1 (r_1 - 1)$$

for any $r_0 + r_1 + 2m = 5$, $r_0 \equiv 0 \pmod{2}$, $r_1 \equiv 1 \pmod{2}$, $\varepsilon_0, \varepsilon_1 = \pm 1$.

3.2 Invariants of del Pezzo surfaces of degree ≥ 2

Starting with the celebrated papers by Mikhalkin [18] and Welschinger [28], the problem of computation and analysis of the behavior of (genus zero) Welschinger invariants of real rational symplectic four-folds, in particular, real del Pezzo surfaces has been addressed in a series of papers (see, e.g., [2, 4, 5, 12, 13, 14, 15, 17, 23, 29]). Some of the techniques developed there apply to computation of higher genus invariants introduced in Section 2. In this section, we demonstrate examples of computations obtained by properly modified methods of [17]. Similarly to [17], we stress on the positivity and asymptotic behavior of our invariants, which particularly yield the existence of real curves of positive genus passing through appropriate real point configurations.

Real del Pezzo surfaces are classified up to deformation equivalence by their degree and the topology of the real point set (see [7]). In degree ≥ 2 , we have the following surfaces X with a disconnected real part: of degree 4 with $\mathbb{R}X \simeq 2S^2$, of degree 3 with $\mathbb{R}X \simeq \mathbb{R}P^2 \amalg S^2$, and of degree 2 with $\mathbb{R}X \simeq \mathbb{R}P^2 \amalg \mathbb{R}P^2$, $(\mathbb{R}P^2 \# \mathbb{R}P^2) \amalg S^2$, $2S^2$, $3S^2$, or $4S^2$, (cf., for instance, [17, Section 5.1]). For all of them, we can define elliptic invariants, for the two last types invariants of genus 2, and for the very last one invariants of genus 3.

Proposition 4 *Let X be a real del Pezzo surface of degree ≥ 2 such that $\mathbb{R}X$ contains (at least) two connected components F_0, F_1 and let $D \in \text{Pic}^{\mathbb{R}}(X)$ be a nef and big divisor class, satisfying relations (1) for $g = 1$. Then the following conditions are satisfied.*

(i) *For any nonnegative integers r_0, r_1 satisfying (2) with $m = 0$ and for any conjugation-invariant class $\varphi \in H_2(X \setminus (F_0 \cup F_1); \mathbb{Z}/2)$, the invariants $W_{1,(r_0,r_1)}(X, D, (F_0, F_1), (1, 1), \varphi)$ do not depend on the choice of the pair r_0, r_1 (thus, further on we omit subindex (r_0, r_1) in the notation).*

(ii) *If X is not of degree 2 with $\mathbb{R}X \simeq 2S^2$, then*

$$W_1(X, D, (F_0, F_1), (1, 1), 0) > 0 , \quad (4)$$

and

$$\lim_{k \rightarrow \infty} \frac{\log W_1(X, kD, (F_0, F_1), (1, 1), 0)}{k \log k} = \lim_{k \rightarrow \infty} \frac{\log \text{GW}_0(X, kD)}{k \log k} = -DK_X . \quad (5)$$

(iii) *If X is of degree 2 with $\mathbb{R}X \simeq 2S^2$, then*

$$W_1(X, D, (F_0, F_1), (1, 1), 0) + W_{1,(-DK_X-1,1)}(X, D, (F_0, F_1), (1, -1), 0) > 0 , \quad (6)$$

and

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{\log \left(W_1(X, kD, (F_0, F_1), (1, 1), 0) + W_{1,(-kDK_X-1,1)}(X, kD, (F_0, F_1), (1, -1), 0) \right)}{k \log k} \\ = \lim_{k \rightarrow \infty} \frac{\log \text{GW}_0(X, kD)}{k \log k} = -DK_X . \end{aligned} \quad (7)$$

Statement (iii) of Proposition 4 can be generalized to genus 2 and 3 invariants of the surfaces X of degree 2 with $\mathbb{R}X \simeq 3S^2$ or $4S^2$:

Proposition 5 (1) *Let X be a real del Pezzo surface of degree 2 with $\mathbb{R}X \simeq 3S^2$ or $4S^2$, F_0, F_1, F_2 three distinct connected components of $\mathbb{R}X$, $D \in \text{Pic}^{\mathbb{R}}(X)$ a nef and big divisor class satisfying relation (1) with $g = 2$, r_0, r_1 odd positive integers satisfying $r_0 + r_1 = -DK_X$. Let $\underline{r}' = (r_0, r_1, 1)$, $\underline{F}' = (F_0, F_1, F_2)$. Then the invariant $W_{2, \underline{r}'}(X, D, \underline{F}', (1, 1, \pm 1)) := W_{2, \underline{r}'}(X, D, \underline{F}', (1, 1, 1), 0) + W_{2, \underline{r}'}(X, D, \underline{F}', (1, 1, -1), 0)$ does not depend on the choice of odd r_0, r_1 subject to $r_0 + r_1 = -DK_X$ (so, further on the subindex \underline{r}' will be omitted), and it satisfies*

$$W_2(X, D, \underline{F}', (1, 1, \pm 1)) > 0$$

and

$$\lim_{k \rightarrow \infty} \frac{\log W_2(X, kD, \underline{F}', (1, 1, \pm 1))}{k \log k} = \lim_{k \rightarrow \infty} \frac{\log \text{GW}_0(X, kD)}{k \log k} = -DK_X.$$

(2) *Let X be a real del Pezzo surface of degree 2 with $\mathbb{R}X \simeq 4S^2$, F_0, F_1, F_2, F_3 be the connected components of $\mathbb{R}X$, and $D \in \text{Pic}^{\mathbb{R}}(X)$ be a nef and big divisor class satisfying relation (1) with $g = 3$, r_0, r_1 odd positive integers satisfying $r_0 + r_1 = -DK_X$. Let $\underline{r}'' = (r_1, r_2, 1, 1)$, $\underline{F}'' = (F_0, F_1, F_2, F_3)$. Then the invariant*

$$W_{3, \underline{r}''}(X, D, \underline{F}'', (1, 1, \pm 1, \pm 1)) := \sum_{\varepsilon_2, \varepsilon_3 = \pm 1} W_{3, \underline{r}''}(X, D, \underline{F}'', (1, 1, \varepsilon_2, \varepsilon_3), 0)$$

does not depend on the choice of odd r_0, r_1 subject to $r_0 + r_1 = -DK_X$ (so, further on the subindex \underline{r}'' will be omitted), and it satisfies

$$W_3(X, D, \underline{F}'', (1, 1, \pm 1, \pm 1)) > 0$$

and

$$\lim_{k \rightarrow \infty} \frac{\log W_3(X, kD, \underline{F}'', (1, 1, \pm 1, \pm 1))}{k \log k} = \lim_{k \rightarrow \infty} \frac{\log \text{GW}_0(X, kD)}{k \log k} = -DK_X.$$

Corollary 2 (1) *Under the hypotheses of Proposition 4(ii) (respectively, 4(iii)), through any generic configuration $\mathbf{w} \in \mathcal{P}_{r_0, r_1, 0}(X, (F_0, F_1))$ (respectively, $\mathbf{w} \in \mathcal{P}_{(-DK_X - 1, 1), 0}(X, (F_0, F_1))$) one can draw a real elliptic curve $C \in |D|$ such that $C \supset \mathbf{w}$.*

(2) *Under the hypotheses of Proposition 5(1) (respectively, 5(2)), through any generic configuration $\mathbf{w} \in \mathcal{P}_{\underline{r}', 0}(X, \underline{F}')$ (respectively, $\mathcal{P}_{\underline{r}'', 0}(X, \underline{F}'')$) one can draw a real curve $C \in |D|$ of genus 2 (respectively, 3) such that $C \supset \mathbf{w}$.*

3.3 Proof of Proposition 4

By blowing up suitable real points, we reduce the consideration to the only surfaces of degree 2. To treat this case, we use real versions of the Abramovich-Bertram-Vakil formula and the Caporaso-Harris-type formulas developed in [17], as well as their direct extensions to elliptic curves. We subsequently prove statements (i), (ii), and (iii).

3.3.1 Proof of statement (i)

Using Theorem 1 and the construction of [17, Sections 4.2 and 5.2], we can assume that X is a generic real fiber of an *elliptic ABV family* (in the terminology of [17, Section 5.2]), which is the following flat, conjugation-invariant family of surfaces $\pi : \mathfrak{X} \rightarrow (\mathbb{C}, 0)$:

- \mathfrak{X} is a smooth three-fold;
- all fibers \mathfrak{X}_t , $t \neq 0$, are del Pezzo of degree 2; the fibers \mathfrak{X}_t , $t \in (\mathbb{R}, 0) \setminus \{0\}$, are real, equivariantly deformation equivalent to X ;
- the central fiber is $\mathfrak{X}_0 = Y \cup Z$, where Y and Z are smooth real surfaces transversally intersecting along a smooth real rational curve E which satisfies $\mathbb{R}E \neq \emptyset$ and is such that (Y, E) is a nodal del Pezzo pair of degree $K_Y^2 = 2$ (i.e., $K_Y E = 0$, $-K_Y C > 0$ for any irreducible curve $C \neq E$, and $(E^2)_Y = -2$, cf. [17, Section 4]), Z is a real quadric surface with $\mathbb{R}Z \simeq S^2$, in which E is a hyperplane section (representing the divisor class $-K_Z/2$),
- $\mathbb{R}E$ divides some connected component F of $\mathbb{R}Y$ into two parts F_+, F_- so that the components $(F_0)_t, (F_1)_t$ of $\mathbb{R}\mathfrak{X}_t$ (corresponding to the given components F_0, F_1 of $\mathbb{R}X$), merge as $t \rightarrow 0$ to F_+ and F_- , respectively.

By [17, Proposition 24], $\text{Pic}^{\mathbb{R}}(X)$ is naturally embedded into $\text{Pic}^{\mathbb{R}}(Y)$ as the orthogonal complement of E . Note also that the given class $\varphi \in H_2(X \setminus (F_0 \cup F_1); \mathbb{Z}/2)$ can be naturally identified with a conjugation-invariant class in $H_2(Y \setminus F; \mathbb{Z}/2)$ (which we denote also by φ).

For a configuration \mathbf{w} of $-DK_X = -DK_Y$ points in F such that r_0 of them lie in F_+ and r_1 other points lie in F_- (we call such a configuration an (r_0, r_1) -*configuration*), denote by $\mathcal{C}_1^{\mathbb{R}}(Y, D, \mathbf{w})$ the set of real elliptic curves $C \in |D|_Y$ passing through \mathbf{w} . By [22, Proposition 2.1], this is a finite set which consists of only immersed curves. Since $DE = 0$, any curve $C \in \mathcal{C}_1^{\mathbb{R}}(Y, D, \mathbf{w})$ has two one-dimensional real branches, in particular, $C \setminus \mathbb{R}C$ splits into two connected components, one of which we denote by $C_{1/2}$. Using [17, Lemma 7], we can replace each nonnodal singular point of any curve $C \in \mathcal{C}_1^{\mathbb{R}}(Y, D, \mathbf{w})$ by its local nodal equigeneric deformation and then correctly define the number

$$W_{1,(r_0,r_1)}(Y, D, F, \varphi, \mathbf{w}) = \sum_{C \in \mathcal{C}_1^{\mathbb{R}}(Y, D, \mathbf{w})} (-1)^{s(C;F) + C_{1/2} \circ \varphi}, \quad (8)$$

where $s(C; F)$ is the number of solitary nodes of C in F .

Lemma 6 *There exists a (r_0, r_1) -configuration \mathbf{w} such that*

$$W_{1,(r_0,r_1)}(X, D, (F_0, F_1), (1, 1), \varphi) = W_{1,(r_0,r_1)}(Y, D, F, \varphi, \mathbf{w}).$$

Proof. Take \mathbf{w} to be a D_0 -CH-configuration in the sense of Appendix 2, where $D_0 \geq D$ is a suitable real effective divisor, and $|\mathbf{w} \cap F_+| = r_0$, $|\mathbf{w} \cap F_-| = r_1$. Extend \mathbf{w} up to a family of configurations $\mathbf{w}_t \subset \mathbb{R}\mathfrak{X}_t$, $t \in (\mathbb{R}, 0)$, and note that each

elliptic curve $C_t \in \mathcal{C}_1^{\mathbb{R}}(\mathfrak{X}_t, D, \mathbf{w}_t)$ degenerates as $t \rightarrow 0$ either to an elliptic curve $C_0 \in \mathcal{C}_1^{\mathbb{R}}(Y, D, \mathbf{w})$, or to the union of an elliptic curve $C'_0 \in \mathcal{C}_1^{\mathbb{R}}(Y, D - mE, \mathbf{w})$, $m > 0$, and $2m$ generating lines of the quadric Z attached to $2m$ intersection points of C'_0 with E (cf. [17, Lemma 22]). However, by Lemma 25, all intersection points of C'_0 with E are real, and hence the above union of the generating lines of Z is not real. Hence the latter degeneration of C_t is not possible, and we are done. \blacksquare

Lemma 7 *If \mathbf{w} is the (r_0, r_1) -configuration from Lemma 6 then*

$$W_{1,(r_0,r_1)}(Y, D, F, \varphi, \mathbf{w}) = W_{Y,E,\varphi+[\mathbb{R}Y \setminus F]}(D - E, 0, 2e_1, 0), \quad (9)$$

where the right-hand side is an ordinary w -number as defined in [17, Section 3.6].

Proof. By construction of Appendix 2, $\mathbf{w} = \{w_i\}_{i \in J}$, where $J \subset \{1, \dots, N\}$, $|J| = r_0 + r_1$. Let $k = \max J$. Consider degenerations of the curves $C \in \mathcal{C}_1^{\mathbb{R}}(Y, D, \mathbf{w})$ induced by the deformation of \mathbf{w} , in which $\mathbf{w}' = \mathbf{w} \setminus \{w_k\}$ stay fixed, and w_k specializes along the arc L_k to the point $z_k \in E$ (see details in Appendix 2). By [22, Proposition 2.6(2)], any curve $C \in \mathcal{C}_1^{\mathbb{R}}(Y, D, \mathbf{w})$ degenerates

- (a) either into the union $C' \cup E$, where $C' \in |D - E|$ is a real immersed elliptic curve, passing through \mathbf{w}' , intersecting E at one point, and having there a smooth branch quadratically tangent to E ,
- (b) or into the union $C'' \cup E$, where $C'' \in |D - E|$ is a real immersed rational curve, passing through \mathbf{w}' and transversally intersecting E in two distinct real points.

By [22, Proposition 2.8(2)], each curve $C' \cup E$ in item (a) gives rise to two curves in $\mathcal{C}_1^{\mathbb{R}}(Y, D, \mathbf{w})$, which are distinguished by (two) deformation patterns given in [22, Lemma 2.10(2)], and which have opposite Welschinger signs (see [23, Proposition 6.1(i)]), and therefore do not contribute to $W_{1,(r_0,r_1)}(Y, D, F, \varphi, \mathbf{w})$. In its turn, each curve $C'' \cup E$ in item (b) gives rise to one curve in $\mathcal{C}_1^{\mathbb{R}}(Y, D, \mathbf{w})$. Furthermore, these curves C'' are counted by the number $W_{Y,E,\varphi+[\mathbb{R}Y \setminus F]}(D, 0, 2e_1, 0)$ with the same signs as the number $W_{1,(r_0,r_1)}(Y, D, F, \varphi, \mathbf{w})$ counts the corresponding deformed curves in $\mathcal{C}_1^{\mathbb{R}}(Y, D, \mathbf{w})$ (cf. the right-hand sides in (8) and [17, Formulas (3) and (4)]), and hence (9) follows. \blacksquare

Statement (i) of Proposition 4 is an immediate consequence of Lemmas 6 and 7.

Remark 8 *Lemmas 6 and 7 allow one to compute all considered invariants $W_1(X, D, (F_0, F_1), (1, 1), \varphi)$ via the recursive formula in [17, Theorem 2]. In Table 1 we present several values of this invariant for $D = -2K_X$ and various real del Pezzo surfaces X (compared with the corresponding Gromov-Witten invariants of genus 1, cf. [4, Examples 4.2 and 6.7]).*

$\deg X$	4	3	2	2	2	2	2
$\mathbb{R}X$	$2S^2$	$\mathbb{R}P^2 \perp\!\!\!\perp S^2$	$2\mathbb{R}P^2$	$(\mathbb{R}P^2 \# \mathbb{R}P^2) \perp\!\!\!\perp S^2$	$2S^2$	$3S^2$	$4S^2$
$W_1(X, -2K)$	112	36	12	12	4	8	16
$GW_1(X, -2K)$	12300	1740	204	204	204	204	204

Table 1: Elliptic invariants of del Pezzo surfaces of degree ≥ 2

3.3.2 Proof of the positivity relation (4)

By Lemmas 6 and 7, to prove (4), it is enough to show that

$$W_{Y,E,[\mathbb{R}Y \setminus F]}(D - E, 0, 2e_1, 0) > 0 . \quad (10)$$

First, we prove an auxiliary inequality. Denote by \mathbb{Z}_+^∞ the semigroup of vectors $\alpha = (\alpha_1, \alpha_2, \dots)$ with countably many nonnegative integer coordinates such that $\|\alpha\| = \sum_i \alpha_i < \infty$, and denote by $\mathbb{Z}_+^{\infty, \text{odd}} \subset \mathbb{Z}_+^\infty$ the subsemigroup of vectors α such that $\alpha_{2i} = 0$ for all $i \geq 0$. By e_1, e_2, \dots we denote the standard unit vectors in \mathbb{Z}_+^∞ . Put $I\alpha = \sum_{i \geq 1} i\alpha_i$ for $\alpha \in \mathbb{Z}_+^\infty$.

Lemma 9 *For any real nodal del Pezzo pair (Y, E) , introduced in Section 3.3.1, any nef divisor class $D' \in \text{Pic}^{\mathbb{R}}(Y)$ such that $D'E \geq 0$ and $-D'K_Y > 0$, and any vectors $\alpha, \beta \in \mathbb{Z}_+^{\infty, \text{odd}}$ such that $I(\alpha + \beta) = D'E$, one has*

$$W_{Y,E,[\mathbb{R}Y \setminus F]}(D', \alpha, \beta, 0) \geq 0 , \quad (11)$$

where $W_{Y,E,[\mathbb{R}Y \setminus F]}(D', \alpha, \beta, 0)$ is an ordinary w -number as defined in [17, Section 3.6].

Proof. For those pairs (Y, E) , which come from real del Pezzo surfaces X with $\mathbb{R}X \simeq S^2 \perp\!\!\!\perp (\mathbb{R}P^2 \# \mathbb{R}P^2)$, $\mathbb{R}P^2 \perp\!\!\!\perp \mathbb{R}P^2$, or $3S^2$, the claim follows from [17, Lemma 39]. Thus, we need to consider the only case of $\mathbb{R}X \simeq 4S^2$. Via the anticanonical map $X \rightarrow \mathbb{P}^2$, the considered surface X is represented as the double covering of \mathbb{P}^2 ramified along a real smooth quartic curve Q_X having four ovals (see Figure 1(a)), whereas $\mathbb{R}X$ doubly covers the four disks bounded by the ovals. In turn, the family \mathcal{X} can be obtained via the blow-up of the node of the double covering of the trivial family $\mathbb{P}^2 \times (\mathbb{C}, 0)$ ramified along an inscribed family of quartics with the nodal central quartic Q_Y shown in Figure 1(b).

To prove (11), we use induction on $R_Y(D', \beta) := -(K_Y + E)D' + \|\beta\| - 1$. The base of induction is provided by [17, Proposition 9(1)], where all nonzero values are equal to 1. For the induction step, we apply the suitably modified formula (6) from [17, Theorem 2(2)]. In the left-hand side of [17, Formula (6)], the summands of the first sum and the factors in the second sum, which correspond to real divisor classes $D^{(i)}$ (in the notation of [17]), are nonnegative by the induction assumption, whereas the factors corresponding to pairs of conjugate divisor classes may be negative. More precisely, these factors correspond to pairs of conjugate (-1) -curves in Y intersecting E . They can be viewed as follows (*cf.* [17, Remark 23]): there are exactly six tangents to the quartic curve Q_Y (Figure 1(b)) passing through the node; they all

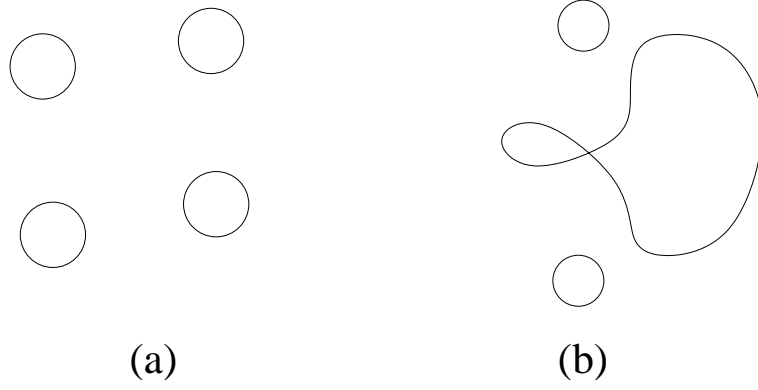


Figure 1: Ramification quartics

are real, and each one is covered by a pair of conjugate (-1) -curves in Y intersecting in a real solitary node, which projects to the tangency point on Q_Y . Thus, a pair of (-1) -curves covering any of the two tangents to the real nodal branch of Q_Y contributes factor (-1) , while a pair of (-1) -curves covering any of the four tangents to the smooth ovals of Q_Y contributes factor 1. Each summand of the second sum in the right-hand side of [17, Formula (6)] can be written as $(l+1)A_mB_{2l+m}$, where all the factors corresponding to pairs of conjugate (-1) -curves are separated in A_m , where m is the number of factors, and the sum of the divisors classes appearing in the remaining part B_{2l+m} equals $D' - E - (2l+m)(K_Y + E)$. By [17, Theorem 2(1g)], any pair of (-1) -curves appears in A_m at most once. Thus, an easy computation converts [17, Formula (6)] into

$$W_{Y,E,[\mathbb{R}Y \setminus F]}(D', \alpha, \beta, 0) = \sum_{j \geq 1, \beta_j > 0} W_{Y,E,[\mathbb{R}Y \setminus F]}(D', \alpha + e_j, \beta - e_j, 0) + B_0 + 2B_1 + B_2,$$

which completes the proof in view of $B_0, B_1, B_2 \geq 0$ (by the induction assumption). ■

Note that $D - E$ is nef (on Y). By [17, Lemma 35(ii)], it is enough to show that $(D - E)E \geq 0$ and $(D - E)E' \geq 0$ for any (-1) -curve E' . We have $(D - E)E = DE - E^2 = 2$. For (-1) -curves disjoint from E , we have $(D - E)E' = DE' \geq 0$ by the nefness of D . Any (-1) -curve E' intersecting E satisfies $E'E = 1$, and hence is not real (any real divisor has even intersection with E , since $[\mathbb{R}E] = 0 \in H_1(\mathbb{R}Y)$). Furthermore, $DE' > 0$. Indeed, otherwise, D would be disjoint both from E' and from its complex conjugate \overline{E}' ; thus, $D(E' + \overline{E}') = D(E + E' + \overline{E}') = 0$, which in view of $\max\{\dim |E' + \overline{E}'|, \dim |E + E' + \overline{E}'|\} = 1$, would contradict the assumption $D^2 > 0$. So, we conclude that $(D - E)E' = DE' - EE' = DE' - 1 \geq 0$.

To complete the proof of (4), we establish a slightly stronger statement than (10).

Lemma 10 *For any real nodal del Pezzo pair (Y, E) of degree ≥ 2 with $\mathbb{R}E \neq \emptyset$ dividing some connected component F of $\mathbb{R}Y$, and any nef divisor class $D' \in \text{Pic}^{\mathbb{R}}(Y)$*

such that $D'E = 2$, one has

$$W_{Y,E,[\mathbb{R}Y \setminus F]}(D', 0, 2e_1, 0) > 0 .$$

Proof. We apply induction on $-D'K_Y$.

By [17, Lemma 35(ii)], D' is nef on X . Since $D' \neq 0$, it is effective on X , and is presented by a smooth curve (see, for instance, [11, Theorems 3, 4, and Remark 3.1.4(B,C)], where the condition $p_a(D) \geq 0$ trivially follows from [11, Formula (3.1.2)]), and hence $-D'K_Y = -D'K_X > 0$. Furthermore, $-D'K_Y \neq 1$. Indeed, otherwise, by the genus formula $(D')^2 \equiv -D'K_Y = 1 \pmod{2}$, that is $(D')^2 \geq 1$, and thus, $p_a(D') \geq 1$. However, $-D'K_X = 1$ and $\dim |K_X| \geq 1$ would imply that a general curve $C \in |D'|_X$ is rational, which is a contradiction. Hence $-D'K_X \geq 2$. Suppose that $-D'K_X = -D'K_Y = 2$. This yields $-D'(K_Y + E) = 0$, which (cf. [17, Lemma 35(iii)]) leaves the only case $K_Y^2 = 2$ and $D' = -K_Y - E$, represented by a smooth rational curve, which finally yields $W_{Y,E,[\mathbb{R}Y \setminus F]}(D', 0, 2e_1, 0) = 1$.

Suppose that $-D'K_Y > 2$. By the genus formula, $(D')^2 > 0$. Then $D'E' > 0$ for any (-1) -curve E' intersecting E (cf. the argument in the proof of the nefness of $D - E$ above). If D' is disjoint from a real (-1) -curve E' such that $E'E = 0$, we blow down E' . If D' is disjoint from a nonreal (-1) -curve E' such that $E'E = 0$, then $E'\overline{E'} = 0$ (since otherwise D' would be disjoint with curves in the one-dimensional linear system $|E' + \overline{E'}|$ contrary to $(D')^2 > 0$), and then we blow down both E' and $\overline{E'}$. After finitely many such steps we arrive to a real nodal del Pezzo surface (Y', E) of degree ≥ 2 and a nef and big divisor class $D' \in \text{Pic}^{\mathbb{R}}(Y')$ such that $D'E = 2$, $-D'K_{Y'} = D'K_Y$, and $D'E' > 0$ for any (-1) -curve in Y' . It follows that $(D' + K_{Y'})E = 2$, and that $D' + K_{Y'}$ nonnegatively intersects any (-1) -curve on Y' . Hence $D' + K_{Y'}$ is nef on Y' . Since $-(D' + K_{Y'})K_{Y'} < -D'K_{Y'} = -DK_Y$, we have $W_{Y',E,[\mathbb{R}Y' \setminus F']}(D' + K_{Y'}, 0, 2e_1, 0) > 0$, where $F' \subset \mathbb{R}Y'$ is the image of F . Then, by [17, Formula (6)] and by Lemma 9,

$$W_{Y,E,[\mathbb{R}Y \setminus F]}(D', 0, 2e_1, 0) = W_{Y',E,[\mathbb{R}Y' \setminus F']}(D', 0, 2e_1, 0)$$

$$\geq W_{Y',E,[\mathbb{R}Y' \setminus F']}(D' + K_{Y'}, 0, 2e_1, 0) \cdot W_{Y',E,[\mathbb{R}Y' \setminus F']}(-K_{Y'} - E, 0, 2e_1, 0) > 0 ,$$

where $W_{Y',E,[\mathbb{R}Y' \setminus F']}(-K_{Y'} - E, 0, 2e_1, 0) = 1$, because $p_a(-K_{Y'} - E) = 0$, and hence a general curve in $|-K_{Y'} - E|_{Y'}$ is smooth rational. \blacksquare

3.4 Proof of the asymptotic relation (5)

It is enough to show that

$$\log W_1(X, kD, (F_0, F_1), (1, 1), 0) \geq (-DK_X)k \log k + O(k) , \quad (12)$$

since by Lemmas 6 and 7, and by [14, Theorem 1],

$$\begin{aligned} \log W_1(X, kD, (F_0, F_1), (1, 1), 0) &= \log W_{Y,E,[\mathbb{R}Y \setminus F]}(kD - E, 0, 2e_1, 0) \\ &\leq \log \text{GW}_0(X, kD) = (-DK_X)k \log k + O(k) . \end{aligned}$$

Using Lemmas 9 and 10, and [17, Formula (6)], we derive for any $k \geq 2$

$$W_*(kD - E, 0, 2e_1, 0) \geq \frac{1}{2} \cdot \frac{(-kDK_Y - 2)!}{(-iDK_Y - 1)!(-(k-i)DK_Y - 1)!}$$

$$\times \sum_{i=1}^{k-1} \left[4 \cdot W_*(iD - E, 0, 2e_1, 0) \cdot W_*((k-i)D - E, 0, 2e_1, 0) \right].$$

where the asterisk stands for the subindex $(Y, E, [\mathbb{R}Y \setminus F])$. This inequality yields that the positive sequence

$$a_n = \frac{W_*(nD - E, 0, 2e_1, 0)}{(-nDK_Y)!}, \quad n \geq 1,$$

satisfies the relation $a_n \geq \lambda \sum_{i=1}^{n-1} a_i$ with some absolute constant $\lambda > 0$. By [17, Lemma 38], $a_n \geq \xi_1 \xi_2^n$, $n \geq 1$, with some positive ξ_1, ξ_2 , which leads to (12).

3.4.1 Proof of statement (iii)

Let X be a real del Pezzo surface of degree 2 with $\mathbb{R}X \simeq 2S^2$ and $D \in \text{Pic}^{\mathbb{R}}(X)$ a nef and big divisor class. So, $F_0 \simeq F_1 \simeq S^2$, and we let $r_0 = -DK_X - 1$, $r_1 = 1$. Since all such surfaces are equivariantly deformation equivalent and in view of Theorem 1(2), we can suppose that X is a fiber \mathfrak{X}'_{τ} , $\tau > 0$, of a flat conjugation-invariant family $\mathfrak{X}' \rightarrow (\mathbb{C}, 0)$ of surfaces, along which the component F_1 collapses to an isolated real nodal point so that in a neighborhood of the node the family is representable as $x_1^2 + x_2^2 + x_3^2 = \tau$. Following [17, Section 4.2], we perform the base change $\tau = t^2$ and blow up the node obtaining finally a conjugation-invariant family (called a 3-*unscrew* $\mathfrak{X} \rightarrow (\mathbb{C}, 0)$ in [17, Section 4.2]) with the central fiber $\mathfrak{X}_0 = Y \cup Z$, where $E = Y \cap Z$ is a smooth real rational curve with $\mathbb{R}E = \emptyset$, (Y, E) being a real nodal del Pezzo pair with $\mathbb{R}Y \simeq S^2$, and Z is a quadric surface in which E represents the divisor class $-K_Z/2$ and which has the real part $\mathbb{R}Z \simeq S^2$. Pick a generic point $w_0 \in \mathbb{R}Z$ and a generic configuration $\mathbf{w}' \subset \mathbb{R}Y$ of $-DK_Y - 1 = -DK_X - 1$ distinct points in $\mathbb{R}Y$, and extend $\{w_0\} \cup \mathbf{w}'$ to smooth equivariant sections $t \mapsto \mathbf{w}_t$ of the family $\mathfrak{X} \rightarrow (\mathbb{C}, 0)$. We can suppose that the curves of the sets $\mathcal{C}_1^{\mathbb{R}}(\mathfrak{X}_t, D, \mathbf{w}_t)$, $t > 0$, form disjoint equisingular families. Their limits at $t = 0$ are as follows.

Lemma 11 *The limit at $t = 0$ of any family $C_t \in \mathcal{C}_1^{\mathbb{R}}(\mathfrak{X}_t, D, \mathbf{w}_t)$, $t > 0$, is a curve $C_0 = C \cup (C' \cup C'')$, where*

- (i) *$C' \subset Y$ is a real rational curve in the linear system $|D - mE|_Y$ for some $m \geq 1$, which passes through \mathbf{w}' and transversally intersects E in m distinct pairs of complex conjugate points,*
- (ii) *the curve $C' \subset Z$ is smooth rational, representing the divisor class $-K_Z/2$, passing through w_1 , and intersecting E at some pair of complex conjugate points of $C \cap E$, the curve C'' consists of $(m-1)$ pairs of complex conjugate lines that generate the two rulings of Z and pass through $(C \cap E) \setminus (C' \cap E)$.*

Furthermore, any curve $C \cup (C' \cup C'')$ as above is a limit of a unique family $C_t \in \mathcal{C}_1^{\mathbb{R}}(\mathfrak{X}_t, D, \mathbf{w}_t)$, $t > 0$.

Proof. The part $C_0 \cap Z$ is a nonempty real curve passing through w_1 . It then belongs to the linear system $|mE|_Z$ for some $m \geq 1$, and hence $C = C_0 \cap Y$ belongs to $|D - mE|_Y$, $m \geq 0$. Since $C \supset \mathbf{w}'$, the dimension count in [22, Proposition 2.1] and the genus bound yield that either C is irreducible of genus 0 or 1, or C consists of two components, one rational and one elliptic. In both cases, the components of C are real and intersect E in pairs of complex conjugate points. Note that C has no elliptic component. Indeed, otherwise, the curve $C_0 \cap Z$ would consist of lines from the rulings of Z and would not hit a generic point $w_0 \in \mathbb{R}Z$, since the family of real elliptic curves in $|D - mE|_Y$ passing through \mathbf{w}' has real dimension one (see [22, Proposition 2.1]). Hence C is real, irreducible, rational, and intersects E in m distinct pairs of complex conjugate points. The asserted structure of $C_0 \cap Z$ follows immediately.

The existence and uniqueness of a family $C_t \in \mathcal{C}_1^{\mathbb{R}}(\mathfrak{X}_t, ((F_0)_t, (F_1)_t), \mathbf{w}_t)$, $t > 0$, with a prescribed limit $C \cup (C' \cup C'')$ satisfying conditions (i), (ii), follow, for instance, from [24, Theorem 2.8]. \blacksquare

Observe that the curves C_t coming from a limit curve $C_0 = C \cup (C' \cup C'')$ with $C \in |D - mE|_Y$ have precisely $m - 1$ solitary nodes in the component $(F_1)_t \subset \mathbb{R}\mathfrak{X}_t$ and no other real nodes. Hence,

$$W_1(X, D, (F_0, F_1), (1, 1), 0) = \sum_{m \geq 1} (-1)^{m-1} 2^{m-1} m W(Y, D - mE, \mathbf{w}') ,$$

$$W_{1,(1,-DK_X-1)}(X, D, (F_0, F_1), (1, -1), 0) = \sum_{m \geq 1} 2^{m-1} m W(Y, D - mE, \mathbf{w}') ,$$

where $W(Y, D - mE, \mathbf{w}') = \sum_C (-1)^{s(C)}$ with C running over all real rational curves in the linear system $|D - mE|_Y$ passing through \mathbf{w}' , and $s(C)$ is the total number of solitary nodes of C . Thus, we obtain

$$\begin{aligned} & W_1(X, D, (F_0, F_1), (1, 1), 0) + W_{1,(1,-DK_X-1)}(X, D, (F_0, F_1), (1, -1), 0) \\ &= \sum_{m \geq 1} 2^{2m-1} (2m - 1) W(Y, D - (2m - 1)E, \mathbf{w}') . \end{aligned}$$

On the other hand, it follows from [17, Theorem 6(2) and Proposition 35] that

$$W(X, D', F_0, [F_1]) = 2 \sum_{m \geq 1} 2^{2m-1} W(Y, D' - (2m - 1)E, \mathbf{w}')$$

for any divisor class $D' \in \text{Pic}^{\mathbb{R}}(X)$, where

$$W(X, D', F_0, [F_1]) = \sum_{C \in \mathcal{C}_0^{\mathbb{R}}(X, D', \mathbf{w}')} (-1)^{s(C; F_0)}$$

is the (rational) Welschinger invariant (in the notation of [17]). So,

$$W_1(X, D, (F_0, F_1), (1, 1), 0) + W_{1,(-DK_X-1,1)}(X, D, (F_0, F_1), (1, -1), 0)$$

$$= \frac{1}{2}W(X, D, F_0, [F_1]) + \sum_{m \geq 1} W(X, D - 2mE, F_0, [F_1]) , \quad (13)$$

and we immediately derive relations (6), (7) from the positivity and asymptotics of Welschinger invariants $W(X, D', F_0, [F_1])$ established in [17, Theorem 7].

3.5 Proof of Proposition 5

Our argument is completely parallel to that in the proof of statement (iii) of Proposition 4 in Section 3.4.1. First, we construct a conjugation-invariant family $\mathfrak{X} \rightarrow (\mathbb{C}, 0)$ of surfaces along which the component F_g (as $g = 2$ or 3) collapses, and X degenerates into the union of a real nodal del Pezzo surface and a quadric surface, intersecting along a real rational curve E with the empty real part. Then, similarly to (13) we derive

$$\begin{aligned} W_{2,\underline{x}'}(X, D, \underline{F}', (1, 1, \pm 1)) &= \frac{1}{2}W_1(X, D, (F_0, F_1), (1, 1), 0) \\ &+ \sum_{m \geq 1} W_1(X, D - 2mE, (F_0, F_1), (1, 1), 0) \end{aligned} \quad (14)$$

and

$$\begin{aligned} W_{3,\underline{x}''}(X, D, \underline{F}'', (1, 1, \pm 1, \pm 1)) &= \frac{1}{2}W_{2,\underline{x}'}(X, D, \underline{F}', (1, 1, \pm 1)) \\ &+ \sum_{m \geq 1} W_{2,\underline{x}'}(X, D - 2mE, \underline{F}', (1, 1, \pm 1), 0) , \end{aligned} \quad (15)$$

provided we establish the following analog of the vanishing statement in [17, Proposition 35]:

Lemma 12 (1) *Let X, D, r_0, r_1 be as in Proposition 5(1). Then*

$$W_1(X, D, (F_0, F_1), (1, 1), [F_2]) = 0 .$$

(2) *Let X, D, r_0, r_1 be as in Proposition 5(2). Then*

$$W_2(X, D, (F_0, F_1, F_2), (1, 1, \varepsilon_2), [F_3]) = 0, \quad \varepsilon_2 = \pm 1 .$$

Observe that formula (14) and Proposition 4(i,ii) yield the first statement of Proposition 5, and subsequently formula (15) yields the second statement of Proposition 5.

Proof of Lemma 12. We prove the first statement; the second one can be proved in the same way.

One can check that the assumption $p_a(D) \geq 2$ yields $-DK_X > 2$, thus, we can assume that $r_1 > 1$. As in Section 3.3.1, we consider an elliptic ABV family $\mathfrak{X} \rightarrow (\mathbb{C}, 0)$ such that the components F_1, F_2 of $X = \mathfrak{X}_t$ ($t > 0$) degenerate into $F \cup \mathbb{R}Z$, where $Y \simeq \mathbb{R}Z \simeq S^2$, $F \setminus \mathbb{R}E = F_+ \cup F_-$, $\mathbb{R}Z \setminus \mathbb{R}E = \mathbb{R}Z_+ \cup \mathbb{R}Z_-$, and we suppose that the limit of F_1 (respectively, F_2) is $F_+ \cup \mathbb{R}Z_+$ (respectively, $F_- \cup \mathbb{R}Z_-$). Then (for an appropriate $D_0 \in \text{Pic}^{\mathbb{R}}(Y)$, $D_0 \geq D$) we choose a D_0 -CH-configuration

\mathbf{w}_0 of $-DK_X$ real points on Y : r_0 points on the component F_0 of $\mathbb{R}Y$ (the limit of F_0) and r_1 points in F_+ . Similarly to Lemma 6, we have

$$W_1(X, D, (F_0, F_1), (1, 1), [F_2]) = W_1(Y, D, (F_0, F_+), (1, 1), \mathbf{w}_0) ,$$

where

$$W_{1,(r_0,r_1)}(Y, D, (F_0, F_+), (1, 1), \mathbf{w}_0) = \sum_{C \in \mathcal{C}_1^{\mathbb{R}}(Y, D, \mathbf{w}_0)} (-1)^{s(C, F_0 \cup F_+)} .$$

As in the proof of Lemma 7, we specialize a suitable point $w \in \mathbf{w}_0 \cap F_+$ to $\mathbb{R}E$, and then each curve $C \in \mathcal{C}_1^{\mathbb{R}}(Y, D, \mathbf{w}_0)$ will degenerate into the union $C' \cup E$, where $C' \in |D - E|$ is a real immersed elliptic curve, passing through $\mathbf{w}' = \mathbf{w}_0 \setminus \{w\}$, intersecting E at one point, and having there a smooth branch quadratically tangent to E (the other option (b) mentioned in the proof of Lemma 7 is not possible, since $C \cap F_-$ is finite). By [22, Proposition 2.8(2)], each curve $C' \cup E$ gives rise to two curves in $\mathcal{C}_1^{\mathbb{R}}(Y, D, \mathbf{w}_0)$, which are distinguished by (two) deformation patterns given in [22, Lemma 2.10(2)], and which have opposite Welschinger signs (see [23, Proposition 6.1(i)]), and therefore do not contribute to $W_{1,(r_0,r_1)}(Y, D, (F_0, F_+), (1, 1), \mathbf{w}_0)$. ■

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4 Appendix A: Degeneration and deformation of curves on rational surfaces

4.1 Moduli spaces of curves

Let Σ be a smooth projective rational surface and $D \in \text{Pic}(\Sigma)$. Denote by $\overline{\mathcal{M}}_{g,n}(\Sigma, D)$, $g \geq 0$, the space of the isomorphism classes of pairs $(\nu : \hat{C} \rightarrow \Sigma, \mathbf{p})$, where \hat{C} is either a Riemann surface of genus g or a connected reducible nodal curve of arithmetic genus g , $\nu_* \hat{C} \in |D|$, $\mathbf{p} = (p_1, \dots, p_n)$ is a sequence of distinct smooth points of \hat{C} , and each component C' of \hat{C} of genus g' , which is contracted by ν ,

contains at least $3 - 2g'$ special points. This moduli space is a projective scheme (see [10]), and there are natural morphisms

$$\Phi_{\Sigma,D} : \overline{\mathcal{M}}_{g,n}(\Sigma, D) \rightarrow |D|, \quad [\nu : \hat{C} \rightarrow \Sigma, \mathbf{p}] \mapsto \nu_* \hat{C} ,$$

$$\text{Ev} : \overline{\mathcal{M}}_{g,n}(\Sigma, D) \rightarrow \Sigma^n, \quad [\nu : \hat{C} \rightarrow \Sigma, \mathbf{p}] \mapsto \nu(\mathbf{p}) .$$

For any subscheme $\mathcal{V} \subset \overline{\mathcal{M}}_{0,n}(\Sigma, D)$, define the *intersection dimension* $\text{idim} \mathcal{V}$ of \mathcal{V} as follows:

$$\text{idim} \mathcal{V} = \dim(\Phi_{\Sigma,D} \times \text{Ev})(\mathcal{V}) ,$$

where the latter value is the maximum over the dimensions of all irreducible components.

Put

$$\begin{aligned} \mathcal{M}_{g,n}^{br}(\Sigma, D) &= \{[\nu : \hat{C} \rightarrow \Sigma, \mathbf{p}] \in \mathcal{M}_{g,n}(\Sigma, D) \mid \hat{C} \text{ is smooth, and} \\ &\quad \nu \text{ is birational on to } \nu(\hat{C})\}, \\ \mathcal{M}_{g,n}^{im}(\Sigma, D) &= \{[\nu : \hat{C} \rightarrow \Sigma, \mathbf{p}] \in \mathcal{M}_{g,n}^{br}(\Sigma, D) \mid \nu \text{ is an immersion}\} . \end{aligned}$$

Denote by $\overline{\mathcal{M}}_{g,n}^{br}(\Sigma, D)$ the closure of $\mathcal{M}_{g,n}^{br}(\Sigma, D)$ in $\overline{\mathcal{M}}_{g,n}(\Sigma, D)$, and introduce also the space

$$\mathcal{M}'_{g,n}(\Sigma, D) = \{[\nu : \hat{C} \rightarrow \Sigma, \mathbf{p}] \in \overline{\mathcal{M}}_{g,n}^{br}(\Sigma, D) \mid \hat{C} \text{ is smooth}\} .$$

Lemma 13 *For any element $[\nu] = [\nu : \hat{C} \rightarrow \Sigma, \mathbf{p}] \in \mathcal{M}_{g,n}^{br}(\Sigma, D)$, the map $\Phi_{\Sigma,D} \times \text{Ev}$ is injective in a neighborhood $[\nu]$, and, for the germ at $[\nu]$ of any irreducible subscheme $\mathcal{V} \subset \mathcal{M}_{0,n}^{br}(\Sigma, D)$, we have*

$$\dim \mathcal{V} = \text{idim} \mathcal{V} .$$

4.2 Curves on del Pezzo and uninodal del Pezzo surfaces

Let Σ be the plane \mathbb{P}^2 blown up at eight distinct points. Denote by \mathcal{D} the Kodaira-Spencer-Kuranishi space of all complex structures on the smooth four-fold Σ , factorized by the action of diffeomorphisms homotopic to the identity. It contains an open dense subset \mathcal{D}^{DP} consisting of del Pezzo surfaces (of degree 1), that is, surfaces with an ample effective anticanonical class. We call a surface $Y \in \mathcal{D}$ uninodal del Pezzo, if it contains a smooth rational (-2) -curve E_Y , and $-K_Y C > 0$ for each irreducible curve $C \neq E_Y$ (in particular, $C^2 \geq -1$). Denote by $\mathcal{D}^{\text{DP}}(A_1) \subset \mathcal{D}$ the subspace formed by uninodal del Pezzo surfaces. Observe that $\mathcal{D}^{\text{DP}}(A_1)$ has codimension 1 in \mathcal{D} , and $\mathcal{D} \setminus (\mathcal{D}^{\text{DP}} \cup \mathcal{D}^{\text{DP}}(A_1))$ is of codimension ≥ 2 in \mathcal{D} .

Through all this section we use the notation

$$n = -DK_{\Sigma} + g - 1 .$$

Lemma 14 *If Σ is a smooth rational surface and $-DK_{\Sigma} > 0$, then the space $\mathcal{M}_{g,0}^{im}(\Sigma, D)$ is either empty, or is a smooth variety of dimension n .*

Proof. Let $[\nu : \hat{C} \rightarrow \Sigma] \in \mathcal{M}_{g,0}^{im}(\Sigma, D)$. The Zariski tangent space to $\mathcal{M}_{g,0}^{im}(\Sigma, D)$ at $[\nu]$ can be identified with $H^0(\hat{C}, \mathcal{N}_{\hat{C}}^\nu)$. Since

$$\deg \mathcal{N}_{\hat{C}}^\nu = -DK_\Sigma + 2g - 2 > 2g - 2 , \quad (16)$$

we have

$$h^1(\hat{C}, \mathcal{N}_{\hat{C}}^\nu) = 0 , \quad (17)$$

and hence $\mathcal{M}_{g,0}^{im}(\Sigma, D)$ is smooth at $[\nu]$ and is of dimension

$$h^0(\hat{C}, \mathcal{N}_{\hat{C}}^\nu) = \deg \mathcal{N}_{\hat{C}}^\nu - g + 1 = -DK_\Sigma + g - 1 = n . \quad (18)$$

■

Lemma 15 (1) Let $\Sigma \in \mathcal{D}^{\text{DP}}$ and $-DK_\Sigma > 0$. Then, the following holds:

- (i) The space $\mathcal{M}_{g,0}^{br}(\Sigma, D)$ is either empty or satisfies $\dim \mathcal{M}_{g,0}^{br}(\Sigma, D) \leq n$.
- (ii) If either $g > 0$ or $D \neq -K_\Sigma$, then $\mathcal{M}_{g,0}^{im}(\Sigma, D) \subset \mathcal{M}_{g,0}^{br,n}(\Sigma, D)$ is an open dense subset, where $\mathcal{M}_{g,0}^{br,n}(\Sigma, D)$ denotes the union of the components of $\mathcal{M}_{g,0}^{br}(\Sigma, D)$ of dimension n .
- (iii) There exists an open dense subset $U^{\text{DP}} \subset \mathcal{D}^{\text{DP}}$ such that, if $\Sigma \in U^{\text{DP}}$, then $\mathcal{M}_{0,0}(\Sigma, -K_\Sigma)$ consists of twelve elements, each corresponding to a rational nodal curve.

(2) There exists an open dense subset $U^{\text{DP}}(A_1) \subset \mathcal{D}(A_1)$ such that if $\Sigma \in U^{\text{DP}}(A_1)$ and $-DK_\Sigma > 0$, then

- (i) $\text{idim} \mathcal{M}'_{g,0}(\Sigma, D) \leq n$;
- (ii) a generic element $[\nu : \hat{C} \rightarrow \Sigma]$ of any irreducible component \mathcal{V} of $\mathcal{M}'_{g,0}(\Sigma, D)$, such that $\text{idim} \mathcal{V} = n$, is an immersion, and the divisor $\nu^*(E_\Sigma)$ consists of DE_Σ distinct points.

Proof. Let $\Sigma \in \mathcal{D}^{\text{DP}} \cup \mathcal{D}^{\text{DP}}(A_1)$. All the statements for the case of an effective $-K_\Sigma - D$ immediately follow from elementary properties of plane lines, conics, and cubics. In particular, a general element of $\mathcal{D}^{\text{DP}} \setminus U^{\text{DP}}$ is the plane blown up at eight generic points on a cuspidal cubic. So, in the sequel we suppose that $-K_\Sigma - D$ is not effective.

In view of Lemma 14, to complete the proof of statements (1) and (2i) it is enough to show that

$$\dim(\mathcal{M}_{g,0}^{br,n}(\Sigma, D) \setminus \mathcal{M}_{g,0}^{im}(\Sigma, D)) < n \quad \text{and} \quad \dim(\mathcal{M}'_{g,0}(\Sigma, D) \setminus \mathcal{M}_{g,0}^{br,n}(\Sigma, D)) < n .$$

Note, first, that, in the case $n = 0$, we have $g = 0$ and $-DK_\Sigma = 1$, and the curves $C \in \Phi_{\Sigma,D}(\mathcal{M}_{g,0}^{br,0}(\Sigma, D))$ are nonsingular due to the bound

$$-DK_\Sigma \geq (C \cdot C')(z) \geq s , \quad (19)$$

coming from the intersection of C with a curve $C' \in |-K_\Sigma|$ passing through a point $z \in C$, where C has multiplicity s . Thus, further on we suppose that $n > 0$.

Let \mathcal{V}_2 be an irreducible component of $\mathcal{M}_{g,0}^{br,n}(\Sigma, D) \setminus \mathcal{M}_0^{im}(\Sigma, D)$, $[\nu : \hat{C} \rightarrow \Sigma] \in \mathcal{V}_2$ a generic element, and let ν have $s \geq 1$ critical points of multiplicities $m_1 \geq \dots \geq m_s \geq 2$. Particularly, bound (19) gives

$$-DK_\Sigma \geq m_1. \quad (20)$$

Then (cf. [6, First formula in the proof of Corollary 2.4]),

$$\dim \mathcal{V}_2 \leq h^0(\hat{C}, \mathcal{N}_{\hat{C}}^\nu / \text{Tors}(\mathcal{N}_{\hat{C}}^\nu)),$$

where the normal sheaf $\mathcal{N}_{\hat{C}}^\nu$ on \hat{C} is defined as the cokernel of the map $d\nu : \mathcal{T}\hat{C} \rightarrow \nu^*\mathcal{T}\Sigma$, and $\text{Tors}(\ast)$ is the torsion sheaf. It follows from [6, Lemma 2.6] (cf. also the computation in [6, Page 363]) that $\deg \text{Tors}(\mathcal{N}_{\hat{C}}^\nu) = \sum_i (m_i - 1)$, and hence

$$\deg \mathcal{N}_{\hat{C}}^\nu / \text{Tors}(\mathcal{N}_{\hat{C}}^\nu) = -DK_\Sigma + 2g - 2 - \sum_{i=1}^s (m_i - 1) \quad (21)$$

which yields

$$\begin{aligned} \dim \mathcal{V}_2 &\leq h^0(\hat{C}, \mathcal{N}_{\hat{C}}^\nu / \text{Tors}(\mathcal{N}_{\hat{C}}^\nu)) \\ &= \max\{\deg \mathcal{N}_{\hat{C}}^\nu / \text{Tors}(\mathcal{N}_{\hat{C}}^\nu) - g + 1, g\} \stackrel{(20)}{\leq} n - (m_1 - 1) < n, \end{aligned} \quad (22)$$

Let \mathcal{V} be an irreducible component of $\mathcal{M}'_{g,0}(\Sigma, D) \setminus \mathcal{M}_{g,0}^{br,n}(\Sigma, D)$. Then a generic element $[\nu : \hat{C} \rightarrow \Sigma] \in \mathcal{V}$ satisfies $\nu_*\hat{C} = sC$ for some $s \geq 2$ and some reduced, irreducible curve $C \subset \Sigma$. It follows from the Riemann-Hirwitz formula, that $g(C) - 1 \leq \frac{1}{s}(g - 1)$, and hence

$$\text{idim} \mathcal{V} \leq -CK_\Sigma + g(C) - 1 \leq -\frac{1}{s}(DK_\Sigma + g - 1) < -DK_\Sigma + g - 1 = n.$$

To complete the proof of (2ii), let us assume that $\dim \mathcal{V} = r$ and the divisor $\nu^*(E_\Sigma)$ contains a multiple point sz , $s \geq 2$. In view of $DE_\Sigma \geq s$ and $(-K_\Sigma - E_\Sigma)D \geq 0$ (remind that D is irreducible and $-K_\Sigma - D$ is not effective), we have $-DK_\Sigma \geq s$. Furthermore, $T_{[\nu]}\mathcal{V}$ can be identified with a subspace of $H^0(\hat{C}, \mathcal{N}_{\hat{C}}^\nu(-(s-1)z))$ (cf. [6, Remark in page 364]). Since

$$\deg \mathcal{N}_{\hat{C}}^\nu(-(s-1)z) = -DK_\Sigma + 2g - 1 - s \stackrel{-DK_\Sigma \geq s}{>} 2g - 2,$$

we have

$$H^1(\hat{C}, \mathcal{N}_{\hat{C}}^\nu(-(s-1)z^*)) = 0,$$

and hence

$$\dim \mathcal{V} \leq h^0(\hat{C}, \mathcal{N}_{\hat{C}}^\nu(-(s-1)z^*)) = n - (s-1) < n$$

contrary to the assumption $\dim \mathcal{V} = n$. ■

Lemma 16 *There exists an open dense subset $V^{\text{DP}} \subset \mathcal{D}^{\text{DP}}$ such that, for each $\Sigma \in V^{\text{DP}}$, the set of effective divisor classes $D \in \text{Pic}(\Sigma)$ satisfying $-DK_\Sigma = 1$ is finite, the set of rational curves in the corresponding linear systems $|D|$ is finite, and any two such rational curves C_1, C_2 either coincide, or are disjoint, or intersect in $C_1 C_2$ distinct points.*

Proof. For the proof see [16, Lemma 10]. ■

Lemma 17 *For each surface $\Sigma \in U^{\text{DP}} \cap V^{\text{DP}}$, each divisor class $D \in \text{Pic}(\Sigma)$ with $-DK_\Sigma > 0$ and $D^2 \geq -1$, and each irreducible component \mathcal{V} of $\overline{\mathcal{M}}_{g,0}^{br,n}(\Sigma, D) \setminus \mathcal{M}_{g,0}^{br,n}(\Sigma, D)$ with $\text{idim} \mathcal{V} = n - 1$, a generic element $[\nu : \hat{C} \rightarrow \Sigma] \in \mathcal{V}$ is such that*

- (i) *either $\hat{C} = \hat{C}_1 \cup \hat{C}_2$ with \hat{C}_1, \hat{C}_2 smooth Riemann surfaces of genera g_1, g_2 , respectively, such that $g = g_1 + g_2$; furthermore, $|\hat{C}_1 \cap \hat{C}_2| = 1$, $[\nu|_{\hat{C}_i} : \hat{C}_i \rightarrow \Sigma] \in \mathcal{M}_{g_i,0}^{im}(\Sigma, D_i)$, where $C_1 = \nu(\hat{C}_1) \neq C_2 = \nu(\hat{C}_2)$, $D_1 D_2 > 0$, and $-D_i K_\Sigma > 0$, $D_i^2 \geq -1$ for each $i = 1, 2$, and, in addition, at any point $z \in C_1 \cap C_2$, any component of (C_1, z) intersects any component of (C_2, z) transversally;*
- (ii) *or $D = -2K_\Sigma$, $g = 0$, $\hat{C} = \hat{C}_1 \cup \hat{C}_2$, $|\hat{C}_1 \cap \hat{C}_2| = 1$, $\nu|_{\hat{C}_1}$ and $\nu|_{\hat{C}_2}$ are immersions of $\hat{C}_1 \simeq \hat{C}_2 \simeq \mathbb{P}^1$ on to the same uninodal curve $C \in |-K_\Sigma|$;*
- (iii) *or $D = -2K_\Sigma$, \hat{C} is a smooth elliptic curve, $\nu : \hat{C} \rightarrow C = \nu(\hat{C})$ is an unramified double covering.*

Furthermore, ν is always an immersion (i.e., a local isomorphism on to the image), and the germ of $\overline{\mathcal{M}}_{g,0}^{br,n}(\Sigma, D)$ at $[\nu]$ is smooth.

Proof. Let \mathcal{V} be an irreducible component of $(\mathcal{M}'_{g,0}(\Sigma, D) \cap \overline{\mathcal{M}}_{g,0}^{br,n}(\Sigma, D)) \setminus \mathcal{M}_{g,0}^{br,n}(\Sigma, D)$ such that $\text{idim} \mathcal{V} = n - 1$ ($\text{idim} \mathcal{V}$ cannot be bigger by Lemma 15(i)). Then its generic element $[\nu : \hat{C} \rightarrow \Sigma]$ is such that $\nu_* \hat{C} = sC$ with a reduced, irreducible C , $s \geq 2$. By the Riemann-Hurwitz formula, $g - 1 = s(g(C) - 1) + \rho/2$, where ρ is the total ramification index of the map $\nu^\vee : \hat{C} \rightarrow C^\vee$, C^\vee being the normalization of C . By Lemma 15(i),

$$\text{idim} \mathcal{V} = n - 1 = -sCK_\Sigma + g - 2 \leq -CK_\Sigma + g(C) - 1 ,$$

which together with the above Riemann-Hurwitz formula yields

$$(s - 1)(-CK_\Sigma + g(C) - 1) + \frac{\rho}{2} \leq 1 .$$

It follows that

- either $s = 2$, $g = g(C) = 1$, $-CK_\Sigma = 1$, $\rho = 0$, and hence $C \in |-K_\Sigma|$, which meets one of the cases in statement (i);

- or $s = 2$, $-CK_\Sigma = 1$, and $g(C) = 0$, which yields $g = 0$ and, in view of the adjunction formula, $C^2 = -1$, or $C^2 \geq 1$; both cases are not possible: the former one is excluded by the assumption $D^2 \geq -1$, whereas the latter one leaves the only option of $C \in |-K_\Sigma|$ a uninodal curve, however, in such a case the map ν cannot be deformed into an element of $\mathcal{M}_{0,0}^{br}(\Sigma, -2K_\Sigma)$, since the deformed map would birationally send \mathbb{P}^1 on to a curve with δ -invariant ≥ 4 in a neighborhood of the node of C , which is bigger than the arithmetic genus, $((-2K_\Sigma)^2 + (-2K_\Sigma)K_\Sigma)/2 + 1 = 2$.

Now let $[\nu : \hat{C} \rightarrow \Sigma]$ be a generic element of an irreducible component \mathcal{V} of $\overline{\mathcal{M}}_{g,0}^{br,n}(\Sigma, D) \setminus \mathcal{M}'_{g,0}(\Sigma, D)$ with $\text{idim} \mathcal{V} = n - 1$. Then \hat{C} has $s \geq 2$ components $\hat{C}_1, \dots, \hat{C}_s$ of genera g_1, \dots, g_s , respectively, and $l \geq s - 1$ nodes. It follows that $g = g_1 + \dots + g_s + l - s + 1$, and, by Lemma 15(i),

$$\text{idim} \mathcal{V} = n - 1 = -DK_\Sigma + g - 2 \leq -DK_\Sigma + (g_1 + \dots + g_s) - s ,$$

and hence $l = 1$, $s = 2$, $g = g_1 + g_2$. By Lemma 15(ii), both $\nu|_{\hat{C}_1}$ and $\nu|_{\hat{C}_2}$ are immersions. Note that the case $\nu(\hat{C}_1) = \nu(\hat{C}_2)$ is possible only when $D_1 = D_2$, $g_1 = g_2$, and $-D_1K_\Sigma + g_1 - 1 = 0$. Since $D_1^2 = D_2^2 \geq 1$ in view of the adjunction formula and the condition $D^2 \geq -1$, we are left with the case $D_1 = D_2 = -K_\Sigma$, and $C = \nu(\hat{C}_1) = \nu(\hat{C}_2) \in |-K_\Sigma|$ a rational curve with the unique node z . The map ν takes the germ (\hat{C}, \hat{z}) isomorphically on to the germ (C, z) , since, otherwise, we would get a deformed map ν with the image whose δ -invariant ≥ 4 , which is bigger than its arithmetic genus, $((-2K_\Sigma)^2 + (-2K_\Sigma)K_\Sigma)/2 + 1 = 2$. Suppose now that $C_1 = \nu(\hat{C}_1) \neq C_2 = \nu(\hat{C}_2)$. Let us show that C_1 and C_2 intersect transversally as proclaimed in statement (i), which would imply that ν is in immersion. If $-D_1K_\Sigma + g_1 - 1 = -D_2K_\Sigma + g_2 - 1 = 0$, then $g_1 = g_2 = 0$ and $-D_1K_\Sigma = -D_2K_\Sigma = 1$, which allows one to apply Lemma 16. If $-D_1K_\Sigma + g_1 - 1 > 0$, then we can vary $\nu|_{\hat{C}_1}$ in $\mathcal{M}_{g_1,0}^{im}(\Sigma, D_1)$ and achieve the required transversality as we did in the proof of Lemma 15(2ii).

At last, the proof of the smoothness of the germ of $\overline{\mathcal{M}}_{g,0}^{br,n}(\Sigma, D)$ at $[\nu]$ literally coincides with that in [16, Lemma 11]. \blacksquare

Lemma 18 *Let $\Sigma \in U^{\text{DP}}$, $g \geq 0$, and $D \in \text{Pic}(\Sigma)$ be an effective divisor class such that $n = -DK_\Sigma + g - 1 \geq 1$. Let $\mathbf{w} = (w_1, \dots, w_n)$ be a sequence of n distinct points in Σ , let σ_i be smooth curve germs in Σ centered at w_i , $n' < i \leq n$, for some $n' < n$, $\mathbf{w}' = (w_i)_{1 \leq i \leq n'}$, and let*

$$\begin{aligned} \overline{\mathcal{M}}_{g,n}^{br}(\Sigma, D; \mathbf{w}', \{\sigma_i\}_{n' < i \leq n}) &= \{[\nu : \hat{C} \rightarrow \Sigma, \mathbf{p}] \in \overline{\mathcal{M}}_{g,n}^{br}(\Sigma, D) : \\ &\quad \nu(p_i) = w_i \text{ for } 1 \leq i \leq n', \nu(p_i) \in \sigma_i, \text{ for } n' < i \leq n\} . \end{aligned}$$

(1) *Suppose that $[\nu : \hat{C} \rightarrow \Sigma, \mathbf{p}]$ either belongs to $\overline{\mathcal{M}}_{g,n}^{br}(\Sigma, D; \mathbf{w}) \cap \mathcal{M}_{g,n}^{im}(\Sigma, D)$, or is as in Lemma 15(iii). If*

$$H^1(\hat{C}, \mathcal{N}_{\hat{C}}^\nu(-\mathbf{p})) = 0 , \tag{23}$$

then Ev sends the germ of $\overline{\mathcal{M}}_{g,n}^{br}(\Sigma, D; \mathbf{w}', \{\sigma_i\}_{n' < i \leq n})$ at $[\nu : \mathbb{P}^1 \rightarrow \Sigma, \mathbf{p}]$ diffeomorphically on to $\prod_{n' < i \leq n} \sigma_i$.

(2) Suppose that $[\nu : \hat{C} \rightarrow \Sigma, \mathbf{p}] \in \overline{\mathcal{M}}_{g,n}^{br}(\Sigma, D; \mathbf{w})$ is such that

- $[\nu : \hat{C} \rightarrow \Sigma] \in \overline{\mathcal{M}}_{g,0}^{br}(\Sigma, D)$ is as in Lemma 17(i),
- $n' \geq -D_1 K_\Sigma + g_1 - 1$, $\#(\mathbf{p} \cap \hat{C}_1) = -D_1 K_\Sigma + g_1 - 1$, $\#(\mathbf{p} \cap \hat{C}_2) = -D_2 K_\Sigma + g_2$, the point sequences $(w_i)_{1 \leq i < -D_1 K_\Sigma}$ and $(w_i)_{-D_1 K_\Sigma \leq i \leq n}$ are generic on the curves $C_1 = \nu_* \hat{C}_1$ and $C_2 = \nu_* \hat{C}_2$, respectively, and the germs σ_i , $n' < i \leq n$, cross C_2 transversally.

Then Ev sends the germ of $\overline{\mathcal{M}}_{g,n}^{br}(\Sigma, D; \mathbf{w}', \{\sigma_i\}_{n' < i \leq n})$ at $[\nu : \hat{C} \rightarrow \Sigma, \mathbf{p}]$ diffeomorphically on to $\prod_{n' < i \leq n} \sigma_i$.

Proof. The first statement follows from the fact that Ev diffeomorphically sends the (smooth by Lemma 14) germ of $\overline{\mathcal{M}}_{g,n}^{br}(\Sigma, D)$ at $[\nu : \hat{C} \rightarrow \Sigma, \mathbf{p}]$ on to the germ of Σ^r at \mathbf{w} . In view of $\dim \overline{\mathcal{M}}_{g,n}^{br}(\Sigma, D) = 2n$ (see Lemma 15(i)) it is sufficient to show that the Zariski tangent space to $\text{Ev}^{-1}(\mathbf{w})$ is zero-dimensional, which is equivalent to

$$h^0(\hat{C}, \mathcal{N}_{\hat{C}}^\nu(-\mathbf{p})) = 0 \quad (24)$$

that in turn immediately follows from (18) and (23).

In the second case, by Lemma 17, the germ of $\overline{\mathcal{M}}_{g,n}^{br}(\Sigma, D)$ at $[\nu : \hat{C} \rightarrow \Sigma, \mathbf{p}]$ is smooth. The general position of the points \mathbf{w} on the curve $C_1 \cup C_2$ yields (23), which similarly to the preceding paragraph suffices for the required diffeomorphism (cf. proof of [16, Lemma 12(2)]). \blacksquare

Consider a proper submersion $\tilde{\Sigma} \rightarrow (\mathbb{C}, 0)$ a smooth three-fold $\tilde{\Sigma}$ such that $\Sigma = \tilde{\Sigma}_0 \in U^{\text{DP}}(A_1)$ and $\tilde{\Sigma}_t \in U^{\text{DP}}$ for all $t \neq 0$. Choose a divisor class $D \in \text{Pic}(\Sigma)$ such that $-DK_\Sigma > 0$ and a nonnegative integer g . Let $\mathbf{w}_t \in \tilde{\Sigma}_t^n$, $t \in (\mathbb{C}, 0)$, be a smooth family of configurations of distinct points such that $\mathbf{w} = \mathbf{w}_0$ is disjoint with the (-2) -curve $E_\Sigma \subset \Sigma$.

Lemma 19 *There exists an open dense subset $\mathcal{U}_n \subset \Sigma^n$ such that, if $\mathbf{w} \in \mathcal{U}_n$, then*

(i) *for any $m \geq 0$ and any element $[\nu : \hat{C} \rightarrow \Sigma, \mathbf{p}] \in \mathcal{M}'_{g,n}(\Sigma, D - mE_\Sigma)$ with $\nu(\mathbf{p}) = \mathbf{w}$, the map ν is an immersion, and the divisor $\nu^*(E_\Sigma) \subset \hat{C}$ consists of $DE_\Sigma + 2m$ distinct points;*

(ii) *each element $[\nu : \hat{C} \cup (\hat{E}_1 \cup \dots \cup \hat{E}_m) \rightarrow \Sigma, \mathbf{p}] \in \overline{\mathcal{M}}_{g,n}(\Sigma, D)$ such that*

- $[\nu : \hat{C} \rightarrow \Sigma, \mathbf{p}] \in \mathcal{M}'_{g,n}(\Sigma, D - mE_\Sigma)$, $\nu(\mathbf{p}) = \mathbf{w}$,
- $\hat{E}_1 \simeq \dots \simeq \hat{E}_m \simeq \mathbb{P}^1$ and ν takes each of $\hat{E}_1, \dots, \hat{E}_m$ isomorphically on to E_Σ ,
- $\hat{E}_i \cap \hat{E}_j = \emptyset$ as $i \neq j$, and $|\hat{C} \cap \hat{E}_i| = 1$ for all $i = 1, \dots, m$,

admits an extension to a smooth family $[\nu_t : \hat{C}_t \rightarrow \tilde{\Sigma}_t, \mathbf{p}_t] \in \overline{\mathcal{M}}_{g,n}(\tilde{\Sigma}_t, D)$, $t \in (\mathbb{C}, 0)$, where $\nu_t(\mathbf{p}_t) = \mathbf{w}_t$ and $[\nu_t : \hat{C}_t \rightarrow \tilde{\Sigma}_t, \mathbf{p}_t] \in \mathcal{M}_{g,n}^{im}(\tilde{\Sigma}_t, D)$,

(iii) the set of families introduced in item (ii) is in one-to-one correspondence with each of the sets $\mathcal{C}_g(\tilde{\Sigma}_t, D, \mathbf{w}_t)$, $t \neq 0$.

Proof. The statement follows from Lemma 15(2) and [26, Theorem 4.2]. \blacksquare

4.3 Deformation of isolated curve singularities

4.3.1 Local equigeneric deformations

Let Σ be a smooth algebraic surface, and z be an isolated singular point of a curve $C \subset \Sigma$. Denote by $J_{C,z} \subset \mathcal{O}_{C,z}$ the Jacobian ideal, and by $J_{C,z}^{cond} \subset \mathcal{O}_{C,z}$ the local conductor ideal, defined as $\text{Ann} \mathcal{O}_{C^\vee} / \mathcal{O}_{C,z}$, where $C^\vee \rightarrow (C, z)$ is the normalization. If $f(x, y) = 0$ is an equation of (C, z) in some local coordinates x, y in (Σ, z) , then

$$J_{C,z} = \langle f_x, f_y \rangle, \quad J_{C,z}^{cond} = \{g \in \mathcal{O}_{C,z} : \text{ord}(g|_{C_i}) \geq \text{ord}(f'|_{C_i}) - \text{mt}(C_i) + 1, i = 1, \dots, m\},$$

where C_1, \dots, C_m are all the components of (C, z) , $f' = \alpha f_x + \beta f_y$ a generic polar, and $\text{mt}(C_i)$ is the intersection number of C_i with a generic smooth line through z (cf. [9, Section 4.2.4]).

Let $B_{C,z}$ be the base of a semiuniversal deformation of the germ (C, z) . This base can be identified with $\mathcal{O}_{C,z} / J_{C,z} \simeq \mathbb{C}^{\tau(C,z)}$, where $J_{C,z} \subset \mathcal{O}_{C,z}$ is the Jacobian ideal, $\tau(C, z)$ the Tjurina number.

Denote by $B_{C,z}^{eg} \subset B_{C,z}$ the equigeneric locus that parameterizes local deformations of (C, z) with the constant δ -invariant equal to $\delta(C, z)$. The following lemma presents the properties of $B_{C,z}^{eg}$, which we will need.

Lemma 20 *The locus $B_{C,z}^{eg}$ is irreducible and has codimension $\delta(C, z)$ in $B_{C,z}$. The subset $B_{C,z}^{eg,im} \subset B_{C,z}^{eg}$ that parameterizes the immersed deformations is open and dense in $B_{C,z}^{eg}$, and consists only of smooth points of $B_{C,z}^{eg}$. The subset $B_{C,z}^{eg,nod} \subset B_{C,z}^{eg}$ that parameterizes the nodal deformations is also open and dense. The complement $B_{C,z}^{eg} \setminus B_{C,z}^{eg,nod}$ is the closure of three codimension-one strata: $B_{C,z}^{eg}(A_2)$ that parameterizes deformations with one cusp A_2 and $\delta(C, z) - 1$ nodes, $B_{C,z}^{eg}(A_3)$ that parameterizes deformations with one tacnode A_3 and $\delta(C, z) - 2$ nodes, and $B_{C,z}^{eg}(D_4)$ that parameterizes deformations with one ordinary triple point D_4 and $\delta(C, z) - 3$ nodes. The tangent cone $T_0 B_{C,z}^{eg}$ (defined as the limit of the tangent spaces at points of $B_{C,z}^{eg,im}$) can be identified with $J_{C,z}^{cond} / J_{C,z}$.*

Proof. The statement follows from [8, Item (iii) in page 435, Theorem 1.4, Theorem 4.15, and Proposition 4.17]. \blacksquare

4.3.2 Local invariance of Welschinger numbers

Suppose now that Σ possesses a real structure, C is a real curve, and z is its real singular point. Let $b \in B_{C,z}^{eg,im}$ be a real point, and let C_b be the corresponding fiber of the semiuniversal deformation of the germ (C, z) . Choose a real point $b' \in B_{C,z}^{eg,nod}$ sufficiently close to b and define Welschinger signs

$$W_b^+ = (-1)^{s_+(C_{b'})}, \quad W_b^- = (-1)^{s_-(C_{b'})},$$

where $s_+(C_{b'})$ (respectively, $s_-(C_{b'})$) is the number of solitary (respectively, non-solitary) nodes of $C_{b'}$.

Lemma 21 *Welschinger signs W_b^+ and W_b^- do not depend on the choice of a real point $b' \in B_{C,z}^{eg,nod}$ sufficiently close to b .*

Proof. Straightforward. ■

Lemma 22 *Let L_t , $t \in (-\varepsilon, \varepsilon) \subset \mathbb{R}$, be a continuous one-parameter family of conjugation-invariant affine subspaces of $B_{C,z}$ of dimension $\delta(C, z)$ such that*

- L_0 passes through the origin and is transversal to $T_0 B_{C,z}^{eg}$,
- $L_t \cap B_{C,z}^{eg} \subset B_{C,z}^{eg,im}$ for each $t \in (-\varepsilon, \varepsilon) \setminus \{0\}$.

Then,

- (i) *the intersection $L_t \cap B_{C,z}^{eg}$ is finite for each $t \in (-\varepsilon', \varepsilon') \setminus \{0\}$, where $\varepsilon' > 0$ is sufficiently small.*
- (ii) *the functions $W^\pm(t) = \sum_{b \in L_t \cap \mathbb{R} B_{C,z}^{eg}} W_b^\pm$ are constant in $(-\varepsilon', \varepsilon') \setminus \{0\}$, where $\varepsilon' > 0$ is sufficiently small.*

Proof. The statement follows from [16, Lemma 15]. ■

4.3.3 Global transversality conditions

If $C \subset \Sigma$ is a curve with isolated singularities, we consider the joint semiuniversal deformation for all singular points of C . The base of this deformation, the equigeneric locus, and the tangent cone to this locus at the point corresponding to C are as follows:

$$B_C = \prod_{z \in \text{Sing}(C)} B_{C,z}, \quad B_C^{eg} = \prod_{z \in \text{Sing}(C)} B_{C,z}^{eg}, \quad T_0 B_C^{eg} = \prod_{z \in \text{Sing}(C)} T_0 B_{C,z}^{eg}.$$

Lemma 23 *Let $[\nu : \hat{C}_1 \rightarrow \Sigma, \mathbf{p}] \in \mathcal{M}_{g,0}^{br}(\Sigma, D)$ and $C = \nu(\hat{C})$. Assume that $n = -DK_\Sigma + g - 1 > 0$. There exists an open dense subset $\mathcal{U} \subset C^n$ such that any $\mathbf{p} \in \mathcal{U}$ consists of n distinct points, the image $\mathbf{w} = \nu(\mathbf{p})$ is an n -tuple of distinct nonsingular points of C , and*

$$H^0(C, \mathcal{J}_C^{cond}(-\mathbf{w}) \otimes \mathcal{O}_\Sigma(D)) = 0. \quad (25)$$

Let \mathbf{w} satisfy (25), $|D|_{\mathbf{w}} \subset |D|$ be the linear subsystem of curves passing through \mathbf{w} , and $\Lambda(\mathbf{w}) \subset B_C$ be the natural image of $|D|_{\mathbf{w}}$.

- (1) *One has $\text{codim}_{B_C} \Lambda(\mathbf{w}) = \dim B_C^{eg}$, and $\Lambda(\mathbf{w})$ intersects $T_0 B_C^{eg}$ transversally.*

- (2) For any n -tuple $\mathbf{w}' \in \Sigma^n$ sufficiently close to \mathbf{w} and such that $\Lambda(\mathbf{w}')$ intersects B_C^{eg} transversally and only at smooth points, the natural map from the germ of $\mathcal{M}_{g,r}(\Sigma, D)$ at $[\nu : \hat{C} \rightarrow \Sigma, \mathbf{p}]$ to B_C^{eg} gives rise to a bijection between the set of elements $[\nu' : \hat{C}' \rightarrow \Sigma, \mathbf{p}']$ such that $\nu'(\mathbf{p}') = \mathbf{w}'$ on one side and $\Lambda(\mathbf{w}') \cap B_C^{eg}$ on the other side.

Proof. The existence of the required set \mathcal{C} immediately follows from the relation

$$h^0(C, \mathcal{J}_C^{cond} \otimes \mathcal{O}_\Sigma(D)) = n, \quad (26)$$

since imposing one by one n generic point constraints, we reduce h^0 to zero. To prove (26) we use the fact that $\mathcal{J}_C^{cond} = \nu_* \mathcal{O}_{\hat{C}}(-\Delta)$, where $\Delta \subset \hat{C}$ is the so-called double-point divisor, whose degree is $\deg \Delta = 2 \sum_{z \in \text{Sing}(C)} \delta(C, z)$ (see, e.g., [6, Section 2.4] or [9, Section 4.2.4]). Thus,

$$\deg(\mathcal{J}_C^{cond} \otimes \mathcal{O}_\Sigma(D)) = D^2 - 2 \sum_{z \in \text{Sing}(C)} \delta(C, z) = -DK_\Sigma + 2g - 2 > 2g - 2,$$

and hence

$$h^1(\hat{C}, \mathcal{J}_C^{cond} \otimes \mathcal{O}_\Sigma(D)) = 0 \quad \text{and} \quad h^0(\hat{C}, \mathcal{J}_C^{cond} \otimes \mathcal{O}_\Sigma(D)) = -DK_\Sigma + 2g - 2 - g + 1 = n.$$

The dimension and the transversality in statement (1) mean that the pull-back of $T_0 B_C^{eg}$ to $|D|$ intersects $|D|_{\mathbf{w}}$ transversally and only at one point, which, in view of the identification of $T_0 B_C^{eg}$ with $\prod_{z \in \text{Sing}(C)} J_{C,z}^{cond} / J_{C,z}$, reduces to (25), since \mathcal{J}_C^{cond} can equivalently be regarded as the ideal sheaf of the zero-dimensional subscheme of C , defined at all singular points $z \in \text{Sing}(C)$ by the local conductor ideals $J_{C,z}^{cond}$.

(2) The second statement of Lemma immediately follows from the first one. ■

5 Appendix B: CH-configurations of points on real uninodal del Pezzo surfaces

Let Σ be a uninodal del Pezzo surface of degree ≥ 2 . Pick an effective divisor class $D \in \text{Pic}(\Sigma)$ represented by a curve not containing E_Σ as a component, and choose integer $g \geq 0$ and two vectors $\alpha, \beta \in \mathbb{Z}_+^\infty$ such that $I(\alpha + \beta) = DE_\Sigma$. Fix a sequence \mathbf{w} of $\|\alpha\|$ distinct points in general position on E_Σ and a positive function $T : \mathbf{w} \rightarrow \mathbb{Z}$ such that $|T^{-1}(i)| = \alpha_i$, $i \geq 1$. Denote by $\mathcal{M}'_{g, \|\alpha\|}(\Sigma, D, \alpha, \beta, \mathbf{w}, T)$ the space of elements $[\nu : \hat{C} \rightarrow \Sigma, \mathbf{p}] \in \mathcal{M}'_{g, \|\alpha\|}(\Sigma, D, \mathbf{w})$ such that

- $\nu^*(\mathbf{w}) = \sum_{p \in \mathbf{p}} T(\nu(p)) \cdot p$,
- $\nu^*(E_\Sigma \setminus \mathbf{w}) = \sum_{q \in \hat{C} \setminus \mathbf{p}} k_q \cdot q$, where the number of the coefficients k_q equal to i is β_i for all $i \geq 1$.

Lemma 24 *If $\mathcal{M}'_{g, \|\alpha\|}(\Sigma, D, \alpha, \beta, \mathbf{w}, T) \neq \emptyset$, then*

- (i) $\text{idim} \mathcal{M}'_{g, \|\alpha\|}(\Sigma, D, \alpha, \beta, \mathbf{w}, T) \leq n = -D(K_\Sigma + E_\Sigma) + g + \|\beta\| - 1$,
- (ii) a general element $[\nu : \hat{C} \rightarrow \Sigma, \mathbf{p}]$ of any component of $\mathcal{M}'_{g, \|\alpha\|}(\Sigma, D, \alpha, \beta, \mathbf{w}, T)$ of intersection dimension n is an immersion, and the curve $C = \nu(\hat{C})$ is nonsingular along E_Σ .

Proof. The statement follows from [22, Proposition 2.1]. ■

Now let Σ be a real uninodal del Pezzo surface with a real (-2) -curve E_Σ such that $\mathbb{R}E_\Sigma \neq \emptyset$. Pick an effective divisor class $D_0 \in \text{Pic}(\Sigma)$, represented by a real curve not containing E_Σ as component, and such that $N = \dim |D_0| > 0$. Denote by $\text{Prec}(D_0)$ the set of real effective divisor classes $D \in \text{Pic}(\Sigma)$, represented by real curves not containing E_Σ as component, and such that $D_0 \geq D$. Notice that $\dim |D| \leq N$.

Let z_1, \dots, z_N be a sequence of N distinct points in general position on $\mathbb{R}E_\Sigma$, and let $z_i(t)$, $t \in [0, 1]$, be a smooth path in $\mathbb{R}\Sigma$ transversal to $\mathbb{R}E_\Sigma$ at $z_i(0) = z_i$, $i = 1, \dots, N$. We shall construct a sequence of points $w_i = z_i(t_i)$, $0 < t_i < 1$, $i = 1, \dots, N$, called a D_0 -CH-configuration (cf. [17, Section 3.5.2]). We perform the construction inductively on $k = 1, \dots, N$. Assume that we have defined t_i , $i < k$, and then construct t_k in the following procedure. Given any data $D \in \text{Prec}(D_0)$, $g \geq 0$, $\alpha, \beta \in \mathbb{Z}_+^\infty$ such that $I(\alpha + \beta) = DE_\Sigma$ and $1 \leq n = -D(K_\Sigma + E_\Sigma) + g + \|\beta\| - 1 \leq k$, and given any subsets $J_1 \subset \{1, \dots, k-1\}$, $J_2 \subset \{k+1, \dots, N\}$ such that $|J_1| = n-1$, $|J_2| = \|\alpha\|$, we impose the following condition:
for $t \in (0, t_k]$, the sets

$$\{[\nu : \hat{C} \rightarrow \Sigma, \mathbf{p}] \in \mathcal{M}'_{g, \|\alpha\|}(\Sigma, D, \alpha, \beta, \{z_i\}_{i \in J_2}, T) \mid w_i \in \nu(\hat{C}), i \in J_1, z_k(t) \in \nu(\hat{C})\}, \quad (27)$$

are finite of a capacity independent of t , and all their elements are presented by immersions. The existence of such $t_k \in (0, 1)$ follows from the fact that there are only finitely many tuples $(D, g, \alpha, \beta, J_1, J_2)$, for which the sets (27), considered for arbitrary $t \in (0, 1)$, are nonempty.

Lemma 25 *In the above notations, let \mathbf{w} be a D_0 -CH-configuration. Suppose that $D \in \text{Prec}(D_0)$, $g \geq 0$, $\alpha, \beta \in \mathbb{Z}_+^\infty$ satisfy $I(\alpha + \beta) = DE_\Sigma$ and $0 \leq n(D, g, \beta) = -D(K_\Sigma + E_\Sigma) + g + \|\beta\| - 1 \leq N$. Then, for any disjoint sets $J_1, J_2 \subset \{1, \dots, N\}$ such that $|J_1| = n(D, g, \beta)$, $|J_2| = \|\alpha\|$, $\max J_1 < \min J_2$, and for any real element of the set*

$$\{[\nu : \hat{C} \rightarrow \Sigma, \mathbf{p}] \in \mathcal{M}'_{g, \|\alpha\|}(\Sigma, D, \alpha, \beta, \{z_i\}_{i \in J_2}, T) \mid w_i \in \nu(\hat{C}), i \in J_1\} \quad (28)$$

the divisor $\nu^(E_\Sigma) \subset \hat{C}$ is supported at only real points.*

Proof. We use induction on $n = n(D, g, \beta)$. The case $n = 0$ necessarily yields $g = 0$ (see [22, Proposition 2.5]), and the desired statement follows then from [17, Lemma 3]. If $n > 0$, we pick $k = \max J_1$ and consider degenerations of a real element of the set (28) in the family $[\nu_t : \hat{C}_t \rightarrow \Sigma, \mathbf{p}_t]$, $t \in (0, t_k]$, corresponding to the specialization of the point w_k to $z_k \in E_\Sigma$ along the arc L_k . By [22, Proposition 2.6], the limit of this family is

- either $[\nu_0 : \hat{C}_0 \rightarrow \Sigma, \mathbf{p}_0] \in \mathcal{M}'_{g, \|\alpha\|+1}(\Sigma, D, \alpha + e_m, \beta - e_m, \{z_i\}_{i \in J_2 \cup \{k\}}, T_0)$, where $T_0|_{J_2} = T$, $T_0(z_k) = m$, which geometrically means that one of the nonfixed intersection points of $\nu(\hat{C})$ with E_Σ of multiplicity m becomes fixed at the position z_i ; the limit element satisfies $n(D, g, \beta - e_m) = n - 1$; hence by the induction assumption all intersection points $\nu_0(\hat{C}_0) \cap E_\Sigma$ are real, and so are the points of $\nu(\hat{C}) \cap E_\Sigma$;
- or $[\nu_0 : \hat{C}_0 \rightarrow \Sigma, \mathbf{p}_0]$ such that \hat{C}_0 splits into components $E_0, \hat{C}_1, \dots, \hat{C}_m$ so that $\nu_0 : E_0 \rightarrow E$ is an isomorphism, the elements $[\nu_0 : \hat{C}_j \rightarrow \Sigma, \mathbf{p}_j] \in \mathcal{M}'_{g_j, \|\alpha^{(j)}\|}(\Sigma, D_j, \alpha^{(j)}, \beta^{(j)}, T_j)$ satisfy $n(D_j, g_j, \beta_j) < n$, $\nu_0(\mathbf{p}_j) \subset \{z_i \mid i \in J_2\}$ for all $j = 1, \dots, m$, and, moreover, the divisor $\nu_t^*(E_\Sigma)$ is supported at the (real) points $\nu_t^{-1}(z_i)$, $1 \leq i \leq N$, and in a slightly deformed proper subset of the set $\nu_0^{-1}(E_\Sigma \setminus \{z_1, \dots, z_N\})$; thus, if $\nu_0 : \hat{C}_j \rightarrow \Sigma$ is real, then by the induction assumption, $\nu_0^{-1}(E_\Sigma) \cap \hat{C}_j$ is real, and hence a small smooth deformation of any of its proper subsets is real too; if $\nu_0 : \hat{C}_j \rightarrow \Sigma$ is not real (particularly, its complex conjugate must be present in the splitting as well), then we have $n = 0$ and $g = 0$, which by [17, Lemma 3] implies that $E_\Sigma \cap \nu_0(\hat{C}_j)$ is just one point z ; furthermore, in the deformation, this node smooths out, and the deformed curve does not intersect E_Σ in a neighborhood of z .

■

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